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Supply Contracts with Financial Hedging

René Caldentey
Stern School of Business, New York University, New York, New York 10012, rcaldent@stern.nyu.edu

Martin B. Haugh
martin.b.haugh@gmail.com

We study the performance of a stylized supply chain where two firms, a retailer and a producer, compete in a Stackelberg game. The retailer purchases a single product from the producer and afterward sells it in the retail market at a stochastic clearance price. The retailer, however, is budget constrained and is therefore limited in the number of units that he may purchase from the producer. We also assume that the retailer's profit depends in part on the realized path or terminal value of some observable stochastic process. We interpret this process as a financial process such as a foreign exchange rate or interest rate. More generally, the process can be interpreted as any relevant economic index. We consider a variation (the flexible contract) of the traditional wholesale price contract that is offered by the producer to the retailer. Under this flexible contract, at $t = 0$ the producer offers a menu of wholesale prices to the retailer, one for each realization of the financial process up to a future time τ. The retailer then commits to purchasing at time τ a variable number of units, with the specific quantity depending on the realization of the process up to time τ. Because of the retailer's budget constraint, the supply chain might be more profitable if the retailer was able to shift some of the budget from states where the constraint is not binding to states where it is binding. We therefore consider a variation of the flexible contract, where we assume that the retailer is able to trade dynamically between zero and τ in the financial market. We refer to this variation as the *flexible contract with hedging*. We compare the decentralized competitive solution for the two contracts with the solutions obtained by a central planner. We also compare the supply chain's performance across the two contracts. We find, for example, that the producer always prefers the flexible contract with hedging to the flexible contract without hedging. Depending on model parameters, however, the retailer might or might not prefer the flexible contract with hedging.

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1. Introduction

We consider the operation of a stylized supply chain with one producer and one retailer. The producer manufactures a single product, which it sells to the retailer. The retailer in turn sells the product in the retail market at a stochastic clearance price. We consider a noncooperative mode of operation in which both players maximize their own profit functions. In particular, we consider a Stackelberg game where the producer, acting as leader, proposes a retail price or menu of prices to the retailer, who then decides how many units to order. As is customary in the supply chain literature (e.g., Lariviere 1998 and Tsay et al. 1998), we are interested in characterizing the solution of the game as well as its efficiency. We measure the efficiency using the so-called *competition penalty*, that is, the ratio of the noncooperative supply chain profits to the centralized supply chain profits (e.g., Cachon and Zipkin 1999).

Our model differs from previous work in two aspects. First, we assume that the retailer operates under a budget constraint. In particular, a limited amount of cash or working capital is available to the retailer for purchasing product units from the producer. Budget constraints are quite common in practice for a number of reasons. For example, many companies have only limited and/or costly access to credit markets. This could be due to (i) high levels of risk aversion in the financial markets, (ii) the presence of asymmetric information whereby management knows that the company is in much better shape than is generally perceived by the markets, or (iii) structural or legal reasons that make it difficult to raise capital. It is worth mentioning that some companies also choose to restrict their managers by imposing budget constraints on their actions.

The imposition of budget constraints has for the most part been ignored in the extensive research on supply chain management. A recent exception is the work by Buzacott and Zhang (2004), where the interplay between inventory decisions and asset-based financing is investigated.

The second distinguishing aspect of our model is the existence of a financial market or economic index whose movements are correlated (we use the term "correlated" loosely in this paper when referring to any form of statistical dependence) with the supply chain's profits. For example, if the producer sells to a foreign retailer and quotes prices in foreign currency units, then his profits, in units of his domestic currency, will be correlated with exchange rate movements. Similarly, if the retailer pays the producer

in arrears, then the producer is exposed to interest rate risk (representing the time value of the delayed payment) as well as possible default risk. It could also be the case that the clearance price for the product in the retail market is influenced in part by the overall state of the economy or the state of particular sectors within the economy. These states might be represented by the value of some well-chosen economic index.

The existence of the financial market affects our framework in two ways. First, the movements of the financial market serve as a public signal that the players can use to negotiate the terms of the procurement contract. Second, the financial market can be used to minimize the impact of the budget constraint. In particular, by trading dynamically in the financial market the retailer can shift resources from states where the budget constraint is not binding to states where it is. It is therefore worth mentioning that the problem managers face is not necessarily one of having too little working capital. Rather, it is the inability to transfer some of that capital from states where it is not needed to states where it is. Even if it is possible and cheap to do so, raising a loan in the financial markets will not solve that problem because it raises the same amount of cash irrespective of whatever state occurs in the future. The ability to trade dynamically in the financial markets, however, does help to address that problem. This ability to shift resources across different states is of particular interest when the two players use the financial market to negotiate the terms of the procurement contract.

As a side remark, we note that in this paper we will use the term "financial market" even when we have a more general economic index in mind. While it is not possible to trade every economic index, many are tradeable. At the Chicago Mercantile Exchange (CME), for example, derivatives can be traded on the consumer price index, nonfarm payrolls, U.S. retail sales, and U.S. jobless claims, among others. Moreover, the current "securitization" trend suggests that ever more economic indices will be tradeable in the future.

The central objective in this paper is to investigate how financial markets and the information they convey can be effectively used in the design of procurement contracts. Our research is motivated by the growing impact that financial markets are having in the operations of nonfinancial corporations. For example, Southwest Airlines has been able to remain profitable in a period when most carriers are struggling to stay in business, by actively trading fuel derivatives to hedge the price they must pay for jet fuel; see Carter et al. (2004). Similarly, manufacturing companies such as Microsoft and GM employ sophisticated trading strategies in the foreign exchange markets to hedge their currency exposure when selling their products abroad (see Chapter 6 in Boyle and Boyle 2001 and Desai and Veblen 2005). Moreover, it is often the case that financial markets impact operating profits in a complex manner. For example, when Microsoft hedges their currency exposure when selling their software in Mexico, they do so in the knowledge that the exchange rate also influences the local demand in Mexico for their software even when the software is priced in Mexican pesos. See Boyle and Boyle (2001). Because the primary purpose of financial markets is to enable the efficient allocation of risk (and resources) among generally risk-averse agents, it is no surprise that an ever-increasing number of nonfinancial risks are being securitized. As this trend continues, the role of financial decision making in operational decision making is sure to grow in significance.

Despite the extensive use of financial risk management by corporations, research in operations management in general (and in supply chain research in particular) has generally borrowed little from the substantial literature in finance and financial engineering. While it is true that various tools from finance such as mean-variance analysis (e.g., Sobel 1994, Chen and Federgruen 2000), portfolio optimization (e.g., Martínez-de-Albéniz and Simchi-Levi 2005), credit risk analysis (e.g., Babich et al. 2007, Buzacott and Zhang 2004), and option pricing theory (e.g., Smith and Nau 1995, Birge 2000) have been applied to certain problems in operations, these problems are generally studied in a setting that does not include the presence of financial markets. Put differently, the tools of finance have been used but the setting of finance has not. There are, of course, some exceptions. In particular, some papers incorporate commodity markets as an integral component of the operational environment. This occurs, for example, in some of the real options and inventory/production control literature (e.g., Brennan and Schwartz 1985, Dixit and Pindyck 1994). However, the incorporation of the financial markets in this work tends to be problem specific and does not generalize easily. Moreover, they do not address the general problem of how companies should dynamically hedge their operating profits and how operational decisions (such as the design of procurement contracts) should interact with these hedging policies.

In this paper, we study how financial markets impact the design and operations of one of the most widely used contracts in practice, the wholesale price contract (Cachon 2003). We will consider three variations of this type of contract that are offered by the producer to the retailer. In the case of the simple contract, the producer offers at time $t = 0$ a fixed wholesale price to the retailer, who then chooses an order quantity. In the case of the flexible contract, the negotiations are also conducted at $t = 0$, but the physical transaction is deferred to a date $\tau > 0$ when the price and order quantity are contingent upon the history of the financial market up to time τ. It is assumed that no trading in the financial markets takes place. The flexible contract with hedging is similar to the flexible contract, except now the retailer has the ability to trade in the financial markets between $t = 0$ and $t = \tau$.

We assume that both players are risk neutral and maximize the *economic value* of their operations, that is, the

expected value of their payoffs under an appropriate equivalent martingale measure (EMM). Because some of the uncertainty in our framework will be driven by nonfinancial noise, the setting of this paper is one of *incomplete markets* (see, for example, Shreve 2004). A standard result from financial economics then implies that a unique EMM will not exist, so an appropriate one would need to be identified using economic principles. We will not concern ourselves with the selection of the appropriate EMM in this paper and will instead assume that it has already been identified. In addition to being economically sound, we will see that using an EMM allows us to model the situation where trading in the financial markets takes place for hedging purposes only and not for speculative purposes. This is consistent with how the financial markets are typically used in practice by nonfinancial corporations. Of course, the ability to trade in the financial markets can and generally does have an indirect impact on the players' profits by expanding the set of feasible order quantities.

The principal contributions of our analysis are (i) we characterize those scenarios in which the presence of financial markets increases the output of the supply chain and the payoffs of both agents, thereby reducing the double marginalization inefficiency; (ii) we show that the producer is always better off if the retailer is able to hedge his budget constraint—the retailer, however, might actually be worse off when he can hedge his budget constraint; (iii) we also show that if the retailer is able to hedge his budget constraint in the financial markets, then it is possible to have a nonoperating supply chain in some circumstances. This contrasts with the case where the retailer does not have access to financial markets when the supply chain is always operative.

The remainder of this paper is organized as follows. Section 2 describes the basic supply chain model and financial market in greater detail. Sections 3 and 4 characterize the solution of the noncooperative game under the flexible contract and the flexible contract with hedging, respectively. To complete the analysis of these contracts, we also compute the centralized solutions and use them to determine the efficiency of the noncooperative supply chain. While the simple contract is the most commonly occurring in practice, it is a special case of the flexible contract with $\tau = 0$, so we do not need to analyze it separately from the flexible contract. We conclude in §5.

2. Model Description

We now describe the model in further detail. We focus first on the supply chain and then consider the financial markets. Finally, we describe the three types of contracts that we analyze in this paper.

2.1. Supply Chain

We model an isolated segment of a competitive supply chain with one producer that produces a single product and one retailer that faces a stochastic clearance price for this product. (Similar models are discussed in detail in §2 of Cachon 2003. See also Lariviere and Porteus 2001.) This clearance price, and the resulting cash flow to the retailer, are realized at a fixed future time $T > 0$. The retailer and producer, however, negotiate the terms of a procurement contract at time $t = 0$. This contract specifies three quantities:

(i) A fixed procurement time τ, with $0 \leqslant \tau \leqslant T$, when the retailer will place a single order. It is worth mentioning that all the results in this paper still go through if τ is allowed to be a *random stopping time*. The problem of choosing the optimal stopping time, τ^* (and other extensions) was formulated and solved in Caldentey and Haugh (2006b).

(ii) A rule that specifies the size of the order, q_τ. Depending on the type of contract under consideration, q_τ might depend on market information available at time τ.

(iii) The payment, $\mathscr{W}(q_\tau)$, that the retailer pays to the producer for fulfilling the order. Again, depending on the type of contract under consideration, $\mathscr{W}(q_\tau)$ might depend on market information available at time τ. The timing of this payment is not important because we shall assume that interest rates are identically zero.

We will restrict ourselves to transfer payments that are linear on the ordering quantity, the so-called *wholesale price contract*, with $\mathscr{W}(q) = wq$, where w is the per-unit wholesale price charged by the producer. We also assume that during the negotiation of the contract, the producer acts as a Stackelberg leader. That is, for a fixed procurement time τ, the producer moves first and proposes a wholesale price,w_τ, to which the retailer then reacts by selecting the ordering level q_τ.

We assume that the producer has unlimited production capacity and that production takes place at time τ with a constant per-unit production cost equal to c_τ. We assume that c_τ is increasing in τ, thereby reflecting the fact that it is generally costly to delay production. The producer's payoff as a function of the wholesale price, w_τ, and the ordering quantity, q_τ, is given by

$$\Pi_{\mathrm{P}} := (w_\tau - c_\tau)q_\tau. \tag{1}$$

We assume that the retailer is restricted by a budget constraint that limits his ordering decisions. In particular, we assume that the retailer has an initial budget B that might be used to purchase product units from the producer. Depending on the type of contract under consideration, the retailer might be able to trade in the financial market during the time interval $[0, \tau]$, thereby transferring cash resources from states where they are not needed to states where they are.

For a given order quantity, q_τ, the retailer collects a random revenue at time T. We compute this revenue using a linear *clearance price model*. That is, given an ordering quantity, q_τ, the market price at which the retailer sells

(clears) these units is a random function, $A - \xi q_\tau$, where A is a nonnegative random variable and ξ is a positive constant. The random variable A models the market size that we assume is unknown while the fixed parameter, ξ, captures the demand elasticity that we assume is known. The retailer's payoff, as a function of w_τ and q_τ, then takes the form

$$\Pi_R := (A - \xi q_\tau) q_\tau - w_\tau q_\tau. \tag{2}$$

We have chosen to use a stochastic clearance price formulation for the following reason. Our goal in this paper is to highlight the benefits of using financial markets in the context of a simple supply chain model. With this objective in mind, we would like to use a formulation that simultaneously captures the stochastic nature of the retailer's payoff and at the same time allows us to clearly isolate the impact that financial markets have on the supply chain performance. A clearance price approach is better suited to achieving this objective than say the newsvendor type of formulation that is commonly encountered in the supply chain literature (see Cachon 2003 for a recent review of supply chain contract models). Moreover, it is easily justified because in practice, unsold units are generally liquidated using secondary markets at discount prices. Therefore, we can view our clearance price as the average selling price across all units and markets.

As stated earlier, depending on the type of contract under consideration, w_τ and q_τ can depend on market information available at time τ. Because $\mathscr{W}(q)$, Π_P, and Π_R are functions of w_τ and q_τ, it is also the case that these quantities can depend on market information available at time τ.

2.2. Financial Market

The financial market is modelled as follows. Let X_t denote the time t value of a tradeable security and let $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ be the filtration generated by X_t on a given probability space, $(\Omega, \mathscr{F}, Q)$. It is not the case that $\mathscr{F}_T = \mathscr{F}$ because we assume that the nonfinancial random variable, A, is $\mathscr{F}$-measurable but not $\mathscr{F}_T$-measurable. We also assume that there is a risk-less *cash account* available in which cash can be deposited. We assume without loss of generality that the interest rate on the cash account is identically equal to zero. Then, the time τ gain (or loss), $G_\tau(\theta)$, that results from following a self-financing $\mathscr{F}_t$-predictable trading strategy, θ_t, can be represented as a stochastic integral with respect to X. (In words, a trading strategy is self-financing if cash is neither deposited with or withdrawn from the portfolio during the trading interval, $[0, T]$. In particular, trading gains or losses are due to changes in the values of the traded securities. See Shreve 2004 for a technical definition of the self-financing property.) For example, in a continuous-time setting, we have

$$G_\tau(\theta) := \int_0^\tau \theta_s \, dX_s, \tag{3}$$

where θ_s represents the number of units of the tradeable security held at time s. The self-financing property then implicitly defines the position at time s in the cash account. In a discrete-time setting, we have

$$G_\tau(\theta) := \sum_{i=0}^{\tau-1} \theta_i (X_{i+1} - X_i). \tag{4}$$

Because we have assumed that interest rates are identically zero, there is no term in (3) or (4) corresponding to gains or losses from the cash account holdings.

We assume that Q is an EMM so that discounted security prices are Q-martingales. Because we have assumed that interest rates are identically zero, however, it is therefore the case that X_t is a Q-martingale. Subject to integrability constraints on the set of feasible trading strategies, we also see that $G_t(\theta)$ is a Q-martingale for every $\mathscr{F}_t$-predictable self-financing trading strategy, θ_t.

Our analysis will be simplified considerably by making a *complete financial markets assumption*. In particular, let G_τ be any suitably integrable contingent claim that is $\mathscr{F}_\tau$-measurable. Then, a complete financial markets assumption amounts to assuming the existence of an $\mathscr{F}_t$-predictable self-financing trading strategy, θ_t, such that $G_\tau(\theta) = G_\tau$. That is, G_τ is *attainable*. This assumption is very common in the financial literature. Moreover, many incomplete financial models can be made complete by simply expanding the set of tradeable securities. When this is not practical, we can simply assume the existence of a market maker with a known pricing function or pricing kernel who is willing to sell G_τ in the marketplace (e.g., Duffie 2001). In this sense, we could then claim that G_τ is indeed attainable. More generally, Duffie may be consulted for further technical assumptions (that we have omitted to specify) regarding the filtration, $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$, feasible trading strategies, etc.

Regardless of how we choose to justify it, assuming complete financial markets simplifies our analysis considerably because, under this assumption, we will never need to solve for a dynamic trading strategy, θ. Instead, we will need only to solve for a contingent claim, G_τ, safe in the knowledge that any such claim is attainable. For this reason, we will drop the dependence of G_τ on θ in the remainder of the paper. The only restriction we will impose on any such trading gain, G_τ, is that the corresponding trading gain process, $G_s := \mathbb{E}_s^{\mathbb{Q}}[G_\tau]$ be a Q-martingale for $s < \tau$. (Whenever we write $\mathbb{E}_s^{\mathbb{Q}}[\cdot]$, it should be understood as $\mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathscr{F}_s]$.) In particular, we will assume that any feasible trading gain, G_τ, satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau] = G_0$, where G_0 is the initial amount of capital that is devoted to trading in the financial market. Without any loss of generality, we will typically assume that $G_0 = 0$. This assumption will be further clarified in §2.3.

A key aspect of our model is the dependence between the payoffs of the supply chain and returns in the financial market. We model this dependence in a parsimonious way

by assuming that returns in the financial market and the random variable A are dependent. We will make the following assumption regarding the conditional distribution of A.

ASSUMPTION 1. *For all* $\tau \in [0, T]$, $\mathbb{E}_\tau^{\mathbb{Q}}[A] \geqslant c_\tau$.

This condition ensures that in any state at time τ, there is a production level, $q \geqslant 0$, for which the retailer's expected market price exceeds the producer's production cost. In particular, this assumption implies that it is possible to profitably operate the supply chain.

2.3. Three Contracts

The final component of our model is the contractual agreement between the producer and the retailer. We consider three different alternatives. Note that in all three cases, the contract itself is negotiated at time $t = 0$, whereas the actual physical transaction takes place at time $\tau \geqslant 0$.

• **Simple Contract (S-Contract):** In the case of the simple contract, the negotiation and physical transaction both take place at the beginning of the planning horizon so that we have $\tau = 0$. In this case, the financial market is not used in the design of the contract, and our model reduces to the traditional wholesale price contract. That is, the producer, acting as a Stackelberg leader, offers a fixed wholesale price, w_0, at time $t = 0$. The retailer, acting as a follower, then determines the quantity, q_0, that he will purchase. The budget constraint in this case takes the form $w_0 q_0 \leqslant B$, where B is the retailer's available budget.

• **Flexible Contract (F-Contract):** In the case of the flexible contract, the physical transaction is postponed to a future date $\tau \in [0, T]$. In this case, the two parties are able to negotiate at time $t = 0$ a contract contingent on the public history, $\mathcal{F}_\tau$, that is available at time τ. Specifically, at time $t = 0$, the producer offers an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, to the retailer. In response to this offer, the retailer decides on an $\mathcal{F}_\tau$-measurable ordering quantity, $q_\tau = q(w_\tau)$. There is a slight abuse of notation here and throughout the paper when we write $q_\tau = q(w_\tau)$. This expression should not be interpreted as implying that q_τ is a function of w_τ because this would imply that q_τ is measurable with respect to the σ-algebra generated by w_τ. However, we require only that q_τ be $\mathcal{F}_\tau$-measurable, so a more appropriate interpretation is to say that $q_\tau = q(w_\tau)$ is the retailer's response to w_τ.

In this, the flexible contract, we assume that the retailer does not hedge his budget constraint by trading in the financial market. Hence, the financial market acts exclusively as a source of public information used to define the terms of the contract. As a result, the budget constraint takes the form

$$w_\tau q_\tau \leqslant B \quad \text{for all } \omega \in \Omega.$$

We note that the S-contract is a special case of the F-contract with $\tau = 0$.

• **Flexible Contract with Hedging (H-Contract):** A flexible contract with hedging is similar to the flexible contract, but now the retailer has access to the financial markets. In particular, the retailer can use the financial market to hedge the budget constraint by purchasing at date $t = 0$ a contingent claim, G_τ, that is realized at date τ and that satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau] = 0$. Given an $\mathcal{F}_\tau$-measurable wholesale price, w_τ, the retailer purchases an $\mathcal{F}_\tau$-measurable contingent claim, G_τ, and selects an $\mathcal{F}_\tau$-measurable ordering quantity, $q_\tau = q(w_\tau)$, to maximize the economic value of his profits. Because of his access to the financial markets, the retailer can weaken the budget constraint, which now becomes

$$w_\tau q_\tau \leqslant B + G_\tau =: B_\tau \quad \text{for all } \omega \in \Omega.$$

Because the no-trading strategy with $G_\tau \equiv 0$ is always an option, it is clear that for a given wholesale price, w_τ, the retailer is always better off by trading in the financial market. Whether or not the retailer will still be better off in equilibrium when he has access to the financial market will be discussed in §4.

By using a flexible contract, the parties postpone their transaction to a future time and in the process improve their estimates of the market clearance price. In this respect, our flexible contracts are very much related to the literature on supply chain contracts with demand forecast updating (e.g., Donohue 2000). In our case, however, the additional information comes from the financial market and its co-dependence with the market clearance price. This feature differs substantially from previous models that normally relate new market information to marketing research and early order commitments (e.g., Azoury 1985, Eppen and Iyer 1997). As well as being a source of information upon which a contract can be based, however, financial markets also enable the players to hedge their cash flows. In particular, the difference between the equilibrium solutions of the F-contract and H-contract will help us quantify the impact that financial trading has on the supply chain performance.

Before proceeding to analyze these contracts, a number of further clarifying remarks are in order.

(1) Our model assumes a common knowledge framework in which all parameters of the model are known to both agents. Because of the Stackelberg nature of the game, this assumption implies that the producer knows the retailer's budget, B, and that both agents use the same EMM to compute the distribution of the market demand. These common knowledge assumptions, although standard in the supply chain management literature, can be too strong in some situations. In this respect, we view our formulation as a starting point for a more realistic (yet complex) model with asymmetric information.

(2) We also make the implicit assumption that the only information available regarding the random variable, A, is what we can learn from the evolution of X_t in the time interval $[0, \tau]$. If this were not the case, then the trading

strategy in the financial market could depend on some nonfinancial information, so it would not be necessary to restrict the trading gain, G_τ, to be $\mathcal{F}_\tau$-measurable. More generally, if Y_t represented some nonfinancial noise that was observable at time t, then the trading strategy, θ_t, would need only to be predictable with respect to the filtration generated by X and Y. In this case, the complete financial market assumption is of less benefit, and it would be necessary for the retailer to solve the considerably harder problem of finding the optimal θ to find the optimal G_τ.

(3) In this model, the producer does not trade in the financial markets because, being risk neutral and not restricted by a budget constraint, he has no incentive to do so. In particular, the Q-martingale property of self-financing trading strategies implies that if the producer devoted an initial capital, F_0, to trading, then we would need to include a term $-F_0+\mathbb{E}_0^{\mathbb{Q}}[F_\tau]$ in his objective function. Here, F_τ denotes the time τ value of the producers's financial portfolio that results from adopting some self-financing trading strategy. However, the Q-martingale property of trading gains implies that this term is identically zero for all such strategies (subject to technical conditions that we mentioned in the previous subsection), so the financial markets provide no benefit to the producer.

(4) A potentially valid criticism of this model is that, in practice, a retailer is often a small entity and might not have the ability to trade in the financial markets. There are a number of responses to this. First, we use the word "retailer" in a loose sense so that it might in fact represent a large entity. For example, an airline purchasing aircraft is a "retailer" that certainly does have access to the financial markets. Second, it is becoming ever cheaper and easier for even the smallest "player" to trade in the financial markets. Finally, even if the retailer does not have access to the financial market, then the producer, assuming he is a big "player," can offer to trade with the retailer or act as his financial broker. As we shall see in §4, it would always be in the producer's interest to do so.

(5) We claimed earlier that, without loss of generality, we could assume that $G_0=0$. This is clear for the following reason. If $G_0=0$, then with a finite initial budget, B, the retailer has a terminal budget of $B_\tau=B+G_\tau$ with which he can purchase product units at time τ and where $\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=0$. If he allocated $a>0$ to the trading strategy, however, then he would have a terminal budget of $B_\tau=B-a+G_\tau$ at time τ but now with $\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=a$. That the retailer is indifferent between the two approaches follows from the fact any terminal budget, B_τ, that is feasible under one modelling approach is also feasible under the other, and vice-versa.

(6) Another potentially valid criticism of this framework is that the class of contracts is too complex. In particular, by insisting only that w_τ is $\mathcal{F}_\tau$-measurable, we are permitting wholesale price contracts that might be too complicated to implement in practice. If this is the case, then we can easily simplify the set of feasible contracts. By using appropriate conditioning arguments, for example, it would be straightforward to impose the tighter restriction that w_τ be $\sigma(X_\tau)$-measurable instead, where $\sigma(X_\tau)$ is the σ-algebra generated by X_τ.

It would also be possible to limit the retailer to a simple class of strategies such as the classic *buy-and-hold strategy*, where at $t=0$ the retailer purchases a fixed number of futures contracts and/or options on the futures contract. This position is then held until time τ, when it is unwound. We expect the results of this paper would also hold under this class of trading strategy, but the analysis would be more cumbersome with no additional insight.

It is also worth noting at this point that the contracts we consider in this paper are all based on public and observable information. This is in contrast to many of the contracts that have been studied in the literature to date that rely on private and not easily verifiable information such as manufacturing costs or product demand. We would therefore argue that the contracts studied in this paper are in fact easier than the traditional contracts to implement in practice.

(7) Finally, we emphasize that this paper does not address mechanism design, i.e., determining the best contract to use, nor does it address how we could achieve full coordination of the supply chain. Instead, we focus on introducing financial markets and market information into the design of the most popular and widely used contract (the wholesale price contract) to improve its efficiency.

We complete this section with a summary of the notation and conventions that will be used throughout the remainder of the paper. The superscripts S, F, and H are used to denote quantities related to the S-contract, F-contract, and H-contract, respectively. The subscripts R, P, and C are used to denote quantities related to the retailer, producer, and central planner, respectively. The subscript τ is used to denote the value of a quantity conditional on time τ information. For example, $\Pi^{\mathrm{H}}_{\mathrm{P}|\tau}$ is the producer's time τ expected payoff under the H-contract. The expected value, $\mathbb{E}_0^{\mathbb{Q}}[\Pi^{\mathrm{H}}_{\mathrm{P}|\tau}]$, is simply denoted by $\Pi^{\mathrm{H}}_{\mathrm{P}}$, and similar expressions hold for the retailer and central planner. Any other notation will be introduced as necessary.

3. Flexible Contract

We now study the F-contract in which the producer offers a wholesale price, w_τ, to the retailer who then selects a corresponding $q_\tau=q(w_\tau)$.

3.1. Decentralized Solution

In response to the wholesale price menu, w_τ, the retailer selects a menu of ordering quantities, $q_\tau=q(w_\tau)$, by solving the following optimization problem:

$$\Pi^{\mathrm{F}}_{\mathrm{R}}(w_\tau)=\mathbb{E}_0^{\mathbb{Q}}\Big[\max_{q_\tau\geqslant 0}\{\mathbb{E}_\tau^{\mathbb{Q}}[(A-\xi q_\tau-w_\tau)q_\tau]\}\Big]$$
$$\text{subject to}\quad w_\tau q_\tau\leqslant B\quad\text{for all } \omega\in\Omega.$$

Note that the expectation inside the max operator is conditional on $\mathcal{F}_\tau$. So for each possible realization of X until time τ, the retailer determines the optimal quantity, q_τ, by solving a procurement problem with wholesale price, w_τ, and budget constraint $w_\tau q_\tau \leqslant B$. The retailer's problem therefore decouples for each such realization of X. Let us define $\bar{A}_\tau := \mathbb{E}_\tau^{\mathbb{Q}}[A]$ and $\bar{A} := \mathbb{E}_0^{\mathbb{Q}}[\bar{A}_\tau]$.

Straightforward calculations show that the solution to the conditional optimization problem is given by

$$q(w_\tau) = \min\left\{\left(\frac{\bar{A}_\tau - w_\tau}{2\xi}\right)^+, \frac{B}{w_\tau}\right\}. \tag{5}$$

The negative effect of the budget constraint on the optimal ordering quantity is clear from (5). Given this, the retailer's best-response strategy, the producer solves

$$\Pi_{\mathrm{P}}^{\mathrm{F}} = \mathbb{E}_0^{\mathbb{Q}}\left[\max_{w_\tau \geqslant c_\tau}\{(w_\tau - c_\tau)q_\tau(w_\tau)\}\right].$$

As was the case with the retailer's problem, the producer's optimization problem decouples for each realization of X until time τ. We use the notation $\Pi_{\mathrm{P}|\tau}^{\mathrm{F}}$ and $\Pi_{\mathrm{R}|\tau}^{\mathrm{F}}$ to denote the payoffs of the producer and retailer, respectively, conditional on $\mathcal{F}_\tau$.

Proposition 1 (Flexible Contract Solution). *Under Assumption* 1, *the equilibrium solution for the flexible contract is*

$$w_\tau^{\mathrm{F}} = \frac{\bar{A}_\tau + \delta_\tau^{\mathrm{F}}}{2} \quad \textit{and} \quad q_\tau^{\mathrm{F}} = \frac{\bar{A}_\tau - \delta_\tau^{\mathrm{F}}}{4\xi}, \tag{6}$$

where

$$\delta_\tau^{\mathrm{F}} := \max\left\{c_\tau, \sqrt{(\bar{A}_\tau^2 - 8\xi B)^+}\right\}.$$

The equilibrium expected payoffs of the players are then given by

$$\Pi_{\mathrm{P}|\tau}^{\mathrm{F}} = \frac{(\bar{A}_\tau + \delta_\tau^{\mathrm{F}} - 2c_\tau)(\bar{A}_\tau - \delta_\tau^{\mathrm{F}})}{8\xi} \quad \textit{and}$$

$$\Pi_{\mathrm{R}|\tau}^{\mathrm{F}} = \frac{(\bar{A}_\tau - \delta_\tau^{\mathrm{F}})^2}{16\xi}. \tag{7}$$

Proof. The proof of this result is straightforward and is therefore omitted.

For notational simplicity, we have not made explicit the dependence of the equilibrium wholesale price, ordering quantity, and players' payoffs on the budget B. We will make this a general rule in this and the following sections.

The auxiliary parameter, δ_τ^{F}, can be interpreted as a modified production cost, greater than or equal to the original cost c_τ that is induced by the budget, B. That is, the state-dependent noncooperative equilibrium in (6) is the same equilibrium that one would obtain if the producer's production cost were δ_τ^{F} and the supplier had an unlimited budget. We can think of this modified cost, δ_τ^{F}, as a negative (random) externality that a limited budget imposes on the entire supply chain. The following is a direct consequence of the previous result.

Corollary 1. *For every $\omega \in \Omega$, the optimal wholesale price, w_τ^{F}, and optimal quantity, q_τ^{F}, are nonincreasing and nondecreasing, respectively, as a function of the budget B. Furthermore,*

$$\lim_{B\downarrow 0} w_\tau^{\mathrm{F}} = \bar{A}_\tau \quad \textit{and} \quad \lim_{B\downarrow 0} q_\tau^{\mathrm{F}} = 0.$$

The corresponding payoffs, $\Pi_{\mathrm{R}|\tau}^{\mathrm{S}}$ and $\Pi_{\mathrm{P}|\tau}^{\mathrm{S}}$, are nondecreasing in B and vanish as $B \downarrow 0$.

Note that the optimal wholesale price, w_τ^{F}, increases as the budget, B, decreases. That is, the more cash-constrained the retailer is, the higher the wholesale price charged by the producer. In fact, the limiting value, $\bar{A}_\tau$, is the maximum price that the producer can charge and still have an operative supply chain; see Equation (5). Note also from Equations (6) and (7) that when the budget is limited, that is $B < B_\tau^{\mathrm{F}} := (\bar{A}_\tau^2 - c_\tau^2)/8\xi$, the wholesale price, ordering quantity, and retailer's payoff are independent of the manufacturing cost c_τ. This threshold, B_τ^{F}, is the budget above which the unconstrained optimal solution is achieved for a given path.

We now compare the equilibrium and the expected profits of the agents as a function of τ. More specifically, we compare the flexible contract where $\tau > 0$ with the simple contract where $\tau = 0$. This comparison is relevant because it reveals the agents' incentives to induce the other party to select one type of contract versus the other. We note that this is not a straightforward comparison because the production costs are different under the two contracts. Let us denote by $\Pi_{\mathrm{P}}^{\mathrm{F}} := \mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{P}|\tau}^{\mathrm{F}}]$ the producer's expected payoff under a flexible contract. Similar notation is used for the retailer, and the superscript "S" will refer to the equilibrium solution of the simple contract.

Proposition 2. *Suppose that $B \leqslant B_\tau^{\mathrm{F}}$ almost surely and $B \leqslant (A^2 - c_0^2)/8\xi$. Then,*

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau^{\mathrm{F}}] \leqslant w^{\mathrm{S}}, \quad \mathbb{E}_0^{\mathbb{Q}}[q_\tau^{\mathrm{F}}] \geqslant q^{\mathrm{S}}, \quad \Pi_{\mathrm{P}}^{\mathrm{F}} \leqslant \Pi_{\mathrm{P}}^{\mathrm{S}}, \quad \textit{and} \quad \Pi_{\mathrm{R}}^{\mathrm{F}} \geqslant \Pi_{\mathrm{R}}^{\mathrm{S}}.$$

Furthermore, in the limit,

$$\lim_{B\downarrow 0} \frac{\Pi_{\mathrm{P}}^{\mathrm{F}}}{\Pi_{\mathrm{P}}^{\mathrm{S}}} = \frac{1}{\bar{A} - c_\tau}\left(\bar{A} - c_\tau \mathbb{E}_0^{\mathbb{Q}}\left[\frac{A}{\bar{A}_\tau}\right]\right) \leqslant 1 \quad \textit{and}$$

$$\lim_{B\downarrow 0} \frac{\Pi_{\mathrm{R}}^{\mathrm{F}}}{\Pi_{\mathrm{R}}^{\mathrm{S}}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\bar{A}}{\bar{A}_\tau}\right] \geqslant 1.$$

However, if $B \geqslant B_\tau^{\mathrm{F}}$ almost surely and $B \geqslant (A^2 - c_0^2)/8\xi$, then

$$\mathbb{E}_0^{\mathbb{Q}}[w_\tau^{\mathrm{F}}] = w^{\mathrm{S}} + \frac{c_\tau - c_0}{2} \quad \textit{and} \quad \mathbb{E}_0^{\mathbb{Q}}[q_\tau^{\mathrm{F}}] = q^{\mathrm{S}} - \frac{c_\tau - c_0}{4\xi}.$$

In addition,

$$\Pi_{\mathrm{P}}^{\mathrm{F}} \geqslant \Pi_{\mathrm{P}}^{\mathrm{S}} \quad \textit{and} \quad \Pi_{\mathrm{R}}^{\mathrm{F}} \geqslant \Pi_{\mathrm{R}}^{\mathrm{S}} \quad \textit{if and only if}$$

$$\mathrm{Var}(\bar{A}_\tau) + c_\tau^2 - c_0^2 \geqslant 2\bar{A}(c_\tau - c_0).$$

Proof. See the appendix.

Proposition 2 compares the supply chain behavior under the simple and flexible contracts as a function of B. If the retailer's budget is small, then the producer is worse off using the F-contract, whereas the retailer is better off.

However, when the budget is large, then the agents' preferences over the contract depend on the additional condition $\mathrm{Var}(\bar{A}_\tau) + c_\tau^2 - c_0^2 \geqslant 2\bar{A}(c_\tau - c_0)$. This condition will be satisfied when the variance $\mathrm{Var}(\bar{A}_\tau)$ is large and/or the cost differential $c_\tau - c_0$ is small.

Proposition 2 provides only a partial characterization of the agents' preferences over the two types of contracts. In particular, the result does not cover those cases in which the budget has an intermediate value that can be greater than B_τ^{F} for some realizations of X (up to time τ) and less than B_τ^{F} for other realizations. In this case, the comparison between the contracts depends on the specific value of B and the distribution of $\bar{A}_\tau$ and must be done on a case-by-case basis. The example of Figure 1 assumes a uniform distribution for $\bar{A}_\tau$. In Case 1 (see the upper set of graphs), the condition $\mathrm{Var}(\bar{A}_\tau) + c_\tau^2 - c_0^2 \geqslant 2\bar{A}(c_\tau - c_0)$ is satisfied, while in Case 2 (see the lower set of graphs), the condition is not satisfied. The graphs on the left show the average wholesale price for the flexible and simple contracts. The graphs in the middle compare the ordering levels, while the graphs on the right plot the ratio of the players' payoffs under the two types of contracts. In Case 1, both players prefer the flexible contract when the budget is large and the reverse conclusion holds in Case 2. Furthermore, when the budget is small, the retailer prefers the F-contract and the producer prefers the S-contract. This observation suggests that the producer is not able to profit from his leadership position in the Stackelberg game when B is small. These observations are consistent with Proposition 2.

3.2. Centralized Solution

To study the efficiency of the noncooperative or decentralized solution, we first need to compute the centralized solution for the flexible contract model. The centralized solution is found by assuming that a central planner, with the same initial budget B, solves

$$\Pi_{\mathrm{C}}^{\mathrm{F}} = \mathbb{E}_0^{\mathbb{Q}}\Big[\max_{q_\tau \geqslant 0}\{\mathbb{E}_\tau^{\mathbb{Q}}[(A - \xi q_\tau - c_\tau)q_\tau]\}\Big] \quad \text{subject to} \quad c_\tau q_\tau \leqslant B \quad \text{for all } \omega \in \Omega.$$

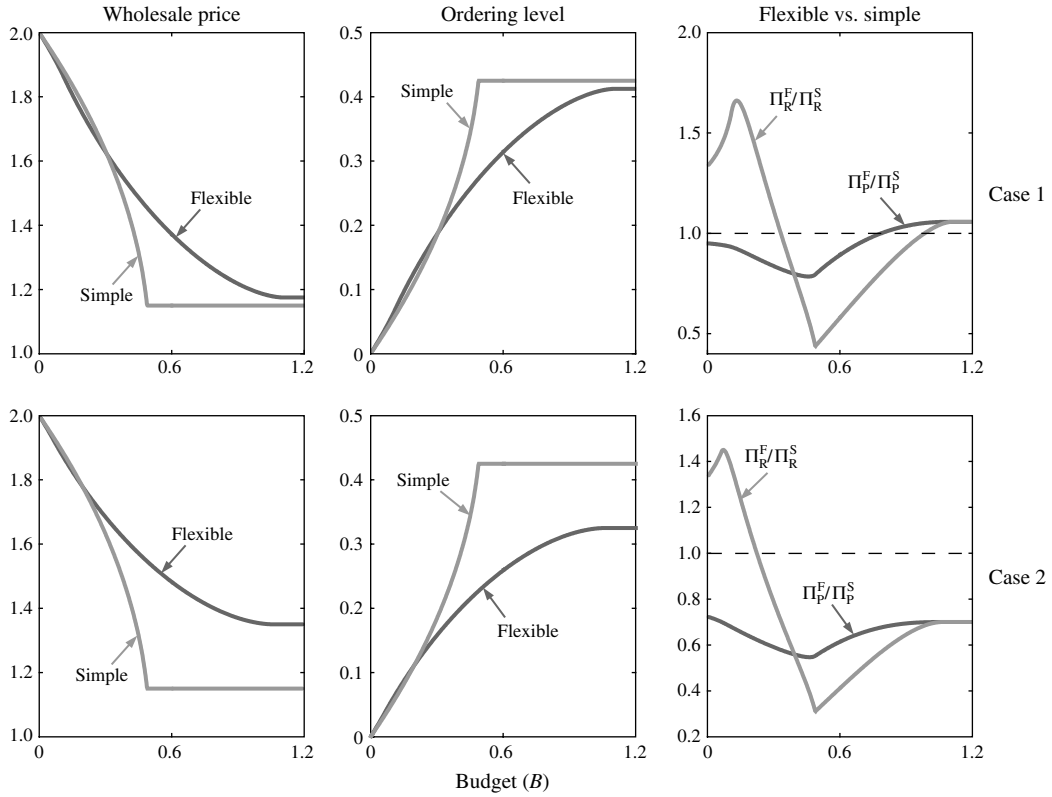
The optimal solution, under Assumption 1, is

$$q_{\mathrm{C}|\tau}^{\mathrm{F}} = \frac{\bar{A}_\tau - \delta_{\mathrm{C}|\tau}^{\mathrm{F}}}{2\xi}, \quad \text{where } \delta_{\mathrm{C}|\tau}^{\mathrm{F}} := \max\left\{c_\tau, \bar{A}_\tau - \frac{2\xi B}{c_\tau}\right\}. \quad (8)$$

Defining $B_{\mathrm{C}|\tau}^{\mathrm{F}} := c_\tau(\bar{A}_\tau - c_\tau)/2\xi$, we obtain that the central planner's expected payoff is given by

$$\Pi_{\mathrm{C}|\tau}^{\mathrm{F}} = \begin{cases} \dfrac{B}{c_\tau^2}\left(c_\tau(\bar{A}_\tau - c_\tau) - \xi B\right) & \text{if } B \leqslant B_{\mathrm{C}|\tau}^{\mathrm{F}}, \\ \dfrac{(\bar{A}_\tau - c_\tau)^2}{4\xi} & \text{if } B \geqslant B_{\mathrm{C}|\tau}^{\mathrm{F}}. \end{cases}$$

Figure 1. Flexible vs. simple contract.



Notes. The conditional demand parameter, $\bar{A}_\tau$, is uniformly distributed in $[1, 3]$. In Case 1, we take $c_\tau = 0.35$, and in Case 2, we take $c_\tau = 0.7$. In both cases, $\xi = 1$, $c_0 = 0.3$.

As was the case with the decentralized solution, the optimal quantity for the centralized solution, $q^{\mathrm{F}}_{\mathrm{C}|\tau}$, is nondecreasing in B and goes to zero as $B \downarrow 0$. The threshold, $B^{\mathrm{F}}_{\mathrm{C}|\tau}$, is the limiting budget above which the centralized solution reaches the unconstrained optimal value, $q^{\mathrm{F}}_{\mathrm{C}|\tau} = (\bar{A}_\tau - c_\tau)/2\xi$.

As was the case with Proposition 2, the following result compares the payoff of the central planner under the simple and flexible contracts.

PROPOSITION 3. *Suppose that $B \leqslant c_\tau(\bar{A}_\tau - c_\tau)/2\xi$ almost surely and $B \leqslant c_0(\bar{A} - c_0)/2\xi$. Then,*

$$\Pi^{\mathrm{F}}_{\mathrm{C}} \geqslant \Pi^{\mathrm{S}}_{\mathrm{C}} \quad \textit{if and only if} \quad (c_\tau^2 - c_0^2)\xi B \geqslant \bar{A} c_0 c_\tau (c_\tau - c_0).$$

However, if $B \geqslant c_\tau(\bar{A}_\tau - c_\tau)/2\xi$ almost surely, then

$$\Pi^{\mathrm{F}}_{\mathrm{C}} \geqslant \Pi^{\mathrm{S}}_{\mathrm{C}} \quad \textit{if and only if} \quad \mathrm{Var}(\bar{A}_\tau) + c_\tau^2 - c_0^2 \geqslant 2\bar{A}(c_\tau - c_0).$$

The proof of Proposition 3 is very similar to the proof of Proposition 2 and is therefore omitted. We see from the first part of the proposition that as $B \downarrow 0$, the central planner prefers the flexible contract. Note that the second part of the proposition is based on the same condition that we derived for the noncooperative game. Therefore, for B sufficiently large, the retailer, the producer, and the central planner either all prefer the flexible contract or all prefer the simple contract.

3.3. Efficiency of the Centralized Solution

Let us now look at the efficiency of the decentralized solution by comparing it to the centralized solution. We first characterize the pathwise efficiency of the F-contract, that is, the efficiency for a given outcome in $\mathcal{F}_\tau$. We will then examine the unconditional efficiency of the contract as perceived at time $t = 0$.

We introduce the following ratios:

$$\mathscr{Q}^{\mathrm{F}}_\tau := \frac{q^{\mathrm{F}}_\tau}{q^{\mathrm{F}}_{\mathrm{C}|\tau}} \quad \text{and} \quad \mathscr{W}^{\mathrm{F}}_\tau := \frac{w^{\mathrm{F}}_\tau}{c_\tau}.$$

The first ratio, $\mathscr{Q}^{\mathrm{F}}_\tau$, measures the degree of inefficiency of the decentralized solution in terms of production output. The second ratio, $\mathscr{W}^{\mathrm{F}}_\tau$, captures the margin over and above the production cost charged by the producer. Naturally, $\mathscr{W}^{\mathrm{F}}_\tau \geqslant 1$, so it follows that $\mathscr{Q}^{\mathrm{F}}_\tau \leqslant 1$. This inefficiency of the decentralized solution has long been recognized in the economics literature and goes under the name of *double marginalization* (e.g., Spengler 1950). We characterize these performance ratios here in the context of a budget constraint.

By Corollary 1, the double marginalization ratio, $\mathscr{W}^{\mathrm{F}}_\tau$, is a nonincreasing function of B and satisfies $\lim_{B\downarrow 0} \mathscr{W}^{\mathrm{F}}_\tau = \bar{A}_\tau/c_\tau$. The ratio, $\mathscr{Q}^{\mathrm{F}}_\tau$, satisfies

$$\mathscr{Q}^{\mathrm{F}}_\tau = \begin{cases} \dfrac{c_\tau\left(\bar{A}_\tau - \sqrt{\bar{A}_\tau^2 - 8\xi B}\right)}{4\xi B} & \text{if } B \leqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \wedge B^{\mathrm{F}}_\tau, \\ \dfrac{\bar{A}_\tau - \sqrt{\bar{A}_\tau^2 - 8\xi B}}{2(\bar{A}_\tau - c_\tau)} & \text{if } B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B \leqslant B^{\mathrm{F}}_\tau, \\ \dfrac{c_\tau(\bar{A}_\tau - c_\tau)}{4\xi B} & \text{if } B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B \leqslant B^{\mathrm{F}}_\tau, \\ 1/2 & \text{if } B \geqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \vee B^{\mathrm{F}}_\tau, \end{cases}$$

where $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

Depending on the values of the average market size, $\bar{A}_\tau$, and production cost, c_τ, either $B^{\mathrm{F}}_{\mathrm{C}|\tau} \geqslant B^{\mathrm{F}}_\tau$ or $B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B^{\mathrm{F}}_\tau$. For this reason, we have to distinguish four possible cases in the computation of $\mathscr{Q}^{\mathrm{F}}_\tau$ as above. It is straightforward to show that $B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B^{\mathrm{F}}_\tau$ if and only if $\bar{A}_\tau \leqslant 3c_\tau$.

The monotonicity of $\mathscr{W}^{\mathrm{F}}_\tau$ implies that $\mathscr{Q}^{\mathrm{F}}_\tau$ increases in B in the range $B \in [0, B^{\mathrm{F}}_{\mathrm{C}|\tau} \wedge B^{\mathrm{F}}_\tau]$. Within this range, smaller budgets therefore hurt the efficiency of the supply chain with respect to the centralized solution more than larger budgets. In the limit, we obtain

$$\lim_{B\downarrow 0} \mathscr{Q}^{\mathrm{F}}_\tau = \frac{c_\tau}{\bar{A}_\tau}.$$

For $B \geqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \vee B^{\mathrm{F}}_\tau$, however, the ratio $\mathscr{Q}^{\mathrm{F}}_\tau$ remains constant at $\frac{1}{2}$.

In the range $B^{\mathrm{F}}_{\mathrm{C}|\tau} \wedge B^{\mathrm{F}}_\tau \leqslant B \leqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \vee B^{\mathrm{F}}_\tau$, the behavior of $\mathscr{Q}^{\mathrm{F}}_\tau$ is different depending on the relationship between $B^{\mathrm{F}}_{\mathrm{C}|\tau}$ and B^{F}_τ. If $B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B^{\mathrm{F}}_\tau$, then $\mathscr{Q}^{\mathrm{F}}_\tau$ is increasing in B. If $B^{\mathrm{F}}_\tau \leqslant B^{\mathrm{F}}_{\mathrm{C}|\tau}$, then $\mathscr{Q}^{\mathrm{F}}_\tau$ is decreasing in B. In both cases, however, the double marginalization inefficiency is minimized at $B = B^{\mathrm{F}}_\tau$.

To analyze the overall efficiency of the F-contract, we look at the *competition penalty*, $\mathscr{P}^{\mathrm{F}}_\tau$ (e.g., Cachon and Zipkin 1999), which is defined as

$$\mathscr{P}^{\mathrm{F}}_\tau := 1 - \left(\frac{\Pi^{\mathrm{F}}_{\mathrm{R}|\tau} + \Pi^{\mathrm{F}}_{\mathrm{P}|\tau}}{\Pi^{\mathrm{F}}_{\mathrm{C}|\tau}}\right).$$

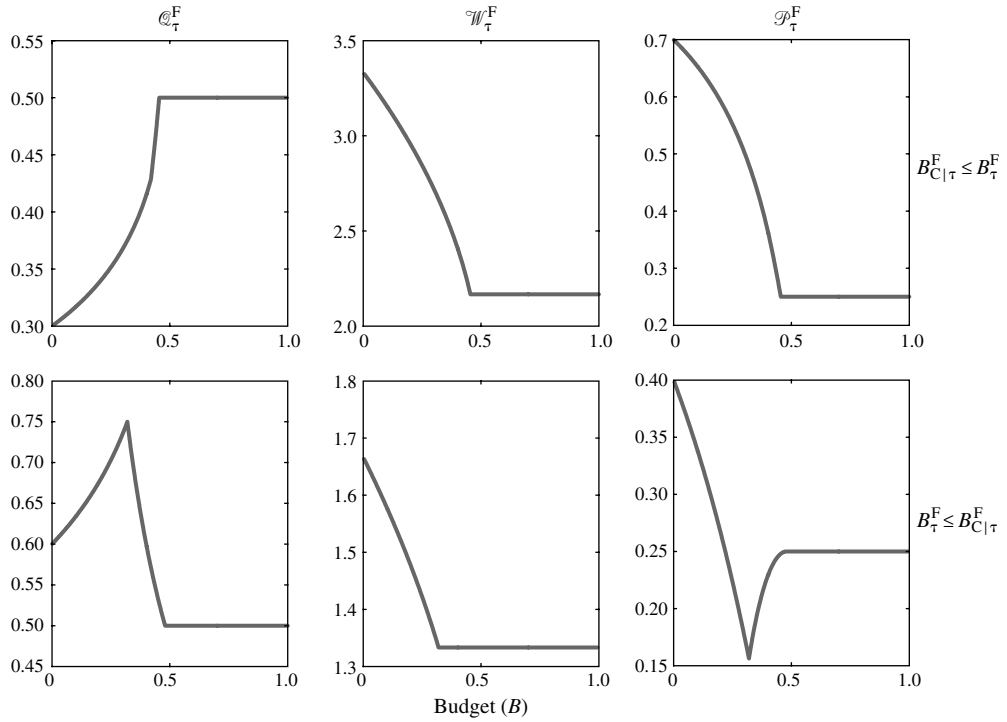
It is clear that $\mathscr{P}^{\mathrm{F}}_\tau \in [0, 1]$ with $\mathscr{P}^{\mathrm{F}}_\tau = 0$, implying that the decentralized chain is perfectly coordinated and achieving the same expected profit as the centralized system. When $\mathscr{P}^{\mathrm{F}}_\tau = 1$, however, the system is completely inefficient. In our setting, we can write the competition penalty as follows:

$$\mathscr{P}^{\mathrm{F}}_\tau = 1 - \left(\frac{\bar{A}_\tau - c_\tau - \xi q^{\mathrm{F}}_\tau}{\bar{A}_\tau - c_\tau - \xi q^{\mathrm{F}}_{\mathrm{C}|\tau}}\right)\mathscr{Q}^{\mathrm{F}}_\tau.$$

PROPOSITION 4. *The competition penalty, as a function of B, is characterized as follows*:

$$\mathscr{P}^{\mathrm{F}}_\tau = \begin{cases} \textit{decreases in } B & \textit{if } B \leqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \wedge B^{\mathrm{F}}_\tau, \\ \textit{decreases in } B & \textit{if } B^{\mathrm{F}}_{\mathrm{C}|\tau} \leqslant B \leqslant B^{\mathrm{F}}_\tau, \\ \textit{increases in } B & \textit{if } B^{\mathrm{F}}_\tau \leqslant B \leqslant B^{\mathrm{F}}_{\mathrm{C}|\tau}, \\ 1/4 & \textit{if } B \geqslant B^{\mathrm{F}}_{\mathrm{C}|\tau} \vee B^{\mathrm{F}}_\tau. \end{cases}$$

Figure 2. $\mathcal{Q}_\tau^{\mathrm{F}}$, $\mathcal{W}_\tau^{\mathrm{F}}$, and $\mathcal{P}_\tau^{\mathrm{F}}$ are plotted against B for the flexible contract.



Notes. The demand model is such that $\bar{A}_\tau = 2$ and $\xi = 1$. The production cost is $c_\tau = 0.6$ for the top row and $c_\tau = 1.2$ for the bottom row.

Proof. The proof is straightforward and is therefore omitted.

Figure 2 summarizes the solution for the F-contract for a given realization in $\mathcal{F}_\tau$. The graphs on the top row correspond to the case $B_{\mathrm{C}|\tau}^{\mathrm{F}} \leqslant B_\tau^{\mathrm{F}}$, while those on the bottom row correspond to $B_\tau^{\mathrm{F}} \leqslant B_{\mathrm{C}|\tau}^{\mathrm{F}}$. The graphs on the left plot the quantity ratio, $\mathcal{Q}_\tau^{\mathrm{F}}$, the graphs in the middle plot the double marginalization ratio, $\mathcal{W}_\tau^{\mathrm{F}}$, and the graphs on the right plot the competition penalty, $\mathcal{P}_\tau^{\mathrm{F}}$. In the case $B_\tau^{\mathrm{F}} \leqslant B_{\mathrm{C}|\tau}^{\mathrm{F}}$, or equivalently $\bar{A}_\tau \leqslant 3c_\tau$, the competition penalty is minimized at $B = B_\tau^{\mathrm{F}}$ and takes the value

$$\mathcal{P}_{\min|X}^{\mathrm{F}} = \frac{(5c_\tau - \bar{A}_\tau)(\bar{A}_\tau - c_\tau)}{(\bar{A}_\tau + c_\tau)(7c_\tau - \bar{A}_\tau)} \leqslant \frac{1}{4}.$$

If $\bar{A}_\tau = c_\tau$, note that the competition penalty vanishes but this is only due to the fact that $q = 0$ for both the decentralized and centralized supply chains.

Thus far, the efficiency of the F-contract has been discussed in a pathwise fashion that is conditional on $\mathcal{F}_\tau$. We now consider the unconditional efficiency. In particular, we are interested in characterizing the expected production efficiency, $\mathcal{Q}^{\mathrm{F}} := \mathbb{E}_0^{\mathbb{Q}}[\mathcal{Q}_\tau^{\mathrm{F}}]$, the expected double marginalization, $\mathcal{W}^{\mathrm{F}} := \mathbb{E}_0^{\mathbb{Q}}[\mathcal{W}_\tau^{\mathrm{F}}]$, and the expected competition penalty, $\mathcal{P}^{\mathrm{F}} := \mathbb{E}_0^{\mathbb{Q}}[\mathcal{P}_\tau^{\mathrm{F}}]$.

The computation of these quantities follows directly from our previous analysis, although the computations are rather tedious due to the number of different cases that arise in terms of B, B_τ^{F}, and $B_{\mathrm{C}|\tau}^{\mathrm{F}}$. The following proposition summarizes the unconditional efficiency of the F-contract in the limiting cases $B \downarrow 0$ and $B \uparrow \infty$.

Proposition 5. *In the limit as the budget, B, goes to zero, we obtain*

$$\lim_{B\downarrow 0} \mathcal{Q}^{\mathrm{F}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{c_\tau}{\bar{A}_\tau}\right] \geqslant \frac{c_\tau}{\bar{A}}, \quad \lim_{B\downarrow 0} \mathcal{W}^{\mathrm{F}} = \frac{\bar{A}}{c_\tau}, \quad \textit{and}$$

$$\lim_{B\downarrow 0} \mathcal{P}^{\mathrm{F}} = 1 - \mathbb{E}^{\mathbb{Q}}\left[\frac{c_\tau}{\bar{A}_\tau}\right] \leqslant \frac{\bar{A} - c_\tau}{\bar{A}}.$$

As $B \to \infty$, we obtain

$$\lim_{B\uparrow\infty} \mathcal{Q}^{\mathrm{F}} = \frac{1}{2}, \quad \lim_{B\uparrow\infty} \mathcal{W}^{\mathrm{F}} = \frac{\bar{A} + c_\tau}{2c_\tau}, \quad \textit{and} \quad \lim_{B\uparrow\infty} \mathcal{P}^{\mathrm{F}} = \frac{1}{4}.$$

Proof. The proof follows from the nonnegativity of $\bar{A}_\tau$, the bounded convergence theorem, and Jensen's inequality. □

Proposition 5 implies that for $B \downarrow 0$ or $B \uparrow \infty$, the expected double marginalization, $\mathcal{W}^{\mathrm{F}}$, decreases with τ. That is, production postponement reduces, on average, the producer's margin. On the other hand, the competition penalty is maximized at $\tau = 0$ for B small and it is constant, independent of τ, for B large.

4. Flexible Contract with Financial Hedging

We now consider the H-contract that is the flexible contract, but where the retailer now has access to the financial markets. The complete financial markets assumption

implies that the retailer can modify his budget by purchasing any $\mathcal{F}_\tau$-measurable financial claim, G_τ, where, as usual, $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is the filtration generated by the financial noise, X_t. Assuming without loss of generality (see §2.3 for details) that an initial capital of zero is devoted to the financial hedging strategy, we then have $\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=0$. The retailer's budget at time τ is then given by $B_\tau = B+G_\tau$. By optimizing over G_τ, the retailer can transfer cash resources from states where the budget constraint is not binding to states where it is. In a partial equilibrium setting, that is for a fixed w_τ, it is clear that the retailer will prefer the H-contract to the F-contract. In our competitive setting, however, this is no longer clear. In fact, we shall see that on some occasions the retailer will prefer the H-contract, but on other occasions he will prefer the F-contract. We shall see that the producer, however, will always prefer the H-contract to the F-contract.

4.1. Decentralized Solution

The sequence of events in the H-contract setting is as follows. At time $t=0$, the producer offers a menu of wholesale prices, w_τ. In response, the retailer selects a menu of ordering quantities, $q_\tau = q(w_\tau)$, as well as an $\mathcal{F}_\tau$-measurable financial claim, G_τ, that satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=0$. At time τ, the outcome is observed and the producer immediately manufactures q_τ product units, which he then sells to the retailer at a per-unit price of w_τ. By construction, the retailer's budget, B_τ, is sufficient to pay the producer for these units. Finally, the retailer sells all the units in the retail market at time T at the stochastic per-unit clearance price, $A-\xi q_\tau$.

The distinguishing feature of the H-contract is that the budget constraint is now a pathwise constraint of the form

$$w_\tau q_\tau \leqslant B_\tau \quad \text{for all } \omega\in\Omega,$$

where $\mathbb{E}_0^{\mathbb{Q}}[B_\tau]=B$. The retailer's problem is then given by

$$\Pi_{\mathrm{R}}^{\mathrm{H}}(w_\tau) = \max_{q_\tau\geqslant 0,\, B_\tau} \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau-\xi q_\tau - w_\tau)q_\tau] \tag{9}$$

$$\text{subject to } w_\tau q_\tau \leqslant B_\tau \quad \text{for all } \omega\in\Omega, \tag{10}$$

$$\mathbb{E}_0^{\mathbb{Q}}[B_\tau]=B. \tag{11}$$

Note that it is no longer possible to decouple the problem and solve it separately for every realization of X (up to time τ), as we did with the F-contract. This is because the new constraint, $\mathbb{E}_0^{\mathbb{Q}}[B_\tau]=B$, binds the entire problem together. We have the following solution to the retailer's problem.

Proposition 6 (Retailer's Optimal Strategy). *Let w_τ be the menu of wholesale prices offered by the producer and let $\mathcal{Q}_\tau$, $\mathcal{X}$ and $\mathcal{X}^c$ be defined as follows*:

$$\mathcal{Q}_\tau := \left(\frac{\bar{A}_\tau - w_\tau}{2\xi}\right)^+, \quad \mathcal{X} := \{\omega\in\Omega\colon B\geqslant \mathcal{Q}_\tau w_\tau\}, \quad \textit{and}$$

$$\mathcal{X}^c := \Omega - \mathcal{X}.$$

The following two cases arise in the computation of the optimal ordering quantity, $q(w_\tau)$, and the financial claim, G_τ.

Case 1. *Suppose that $\mathbb{E}_0^{Q}[\mathcal{Q}_\tau w_\tau]\leqslant B$. Then, $q(w_\tau)=\mathcal{Q}_\tau$, and there are infinitely many choices of the optimal claim, G_τ. One natural choice is to take*

$$G_\tau = [\mathcal{Q}_\tau w_\tau - B]\cdot\begin{cases}\delta & \text{if } \omega\in\mathcal{X},\\ 1 & \text{if } \omega\in\mathcal{X}^c,\end{cases}$$

$$\delta := \frac{\int_{\mathcal{X}^c}[\mathcal{Q}_\tau w_\tau - B]\,\mathrm{d}\mathbb{Q}}{\int_{\mathcal{X}}[B-\mathcal{Q}_\tau w_\tau]\,\mathrm{d}\mathbb{Q}}.$$

In this case (possibly due to the ability to trade in the financial market), the budget constraint is not binding.

Case 2. *Suppose that $B<\mathbb{E}_0^{\mathbb{Q}}[\mathcal{Q}_\tau w_\tau]$. Then,*

$$q_\tau(w_\tau) = \left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+ \quad \textit{and} \quad G_\tau = q_\tau(w_\tau)w_\tau - B,$$

where $\lambda\geqslant 0$ solves

$$\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right] = B.$$

Proof. It is straightforward to see that $\mathcal{Q}_\tau$ is the retailer's optimal ordering level given the wholesale price menu, w_τ, in the absence of a budget constraint. To implement this solution, the retailer would need a budget $\mathcal{Q}_\tau w_\tau$ for all $\omega\in\Omega$. Therefore, if the retailer can generate a financial gain, G_τ, such that $\mathcal{Q}_\tau w_\tau \leqslant B+G_\tau$ for all $\omega\in\Omega$, then he would be able to achieve his unconstrained optimal solution.

By definition, $\mathcal{X}$ contains all those states for which $B\geqslant \mathcal{Q}_\tau w_\tau$. That is, the original budget B is large enough to cover the optimal purchasing cost for all $\omega\in\mathcal{X}$. However, for $\omega\in\mathcal{X}^c$, the initial budget is not sufficient. The financial gain, G_τ, then allows the retailer to transfer resources from $\mathcal{X}$ to $\mathcal{X}^c$.

Suppose that the condition in Case 1 holds so that $\mathbb{E}_0^{\mathbb{Q}}[\mathcal{Q}_\tau w_\tau]\leqslant B$. Note that according to the definition of G_τ in this case, we see that $B+G_\tau = \mathcal{Q}_\tau w_\tau$ for all $\omega\in\mathcal{X}^c$. For $\omega\in\mathcal{X}$, however, $B+G_\tau = (1-\delta)B+\delta\mathcal{Q}_\tau w_\tau \geqslant \mathcal{Q}_\tau w_\tau$. The inequality follows because $\delta\leqslant 1$. G_τ therefore allows the retailer to implement the unconstrained optimal solution. The only point that remains to check is that G_τ satisfies $\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=0$. This follows directly from the definition of δ.

Suppose now that the condition specified in Case 2 holds. We solve the retailer's optimization problem in (9) by relaxing the gain constraint (11) with a Lagrange multiplier, λ. We also relax the budget constraint in (10) for each realization of X up to time τ. The corresponding multiplier for each such realization is denoted by $\beta_\tau\mathrm{d}\mathbb{Q}$, where β_τ plays the role of a Radon-Nikodym derivative of a positive measure that is absolutely continuous with respect to $\mathbb{Q}$. The first-order optimality conditions for the relaxed version of the retailer's problem are then given by

$$q_\tau = \frac{(\bar{A}_\tau - w_\tau(1+\beta_\tau))^+}{2\xi},$$

$$\beta_\tau = \lambda, \quad \beta_\tau(w_\tau q_\tau - B + G_\tau)=0, \quad \beta_\tau\geqslant 0, \quad \text{and}$$

$$\mathbb{E}_0^{\mathbb{Q}}[G_\tau]=0.$$

It is straightforward to show that the solution given in Case 2 of the proposition satisfies these optimality conditions; only the nonnegativity of β_τ needs to be checked separately. To prove this, note that $\beta_\tau = \lambda$; therefore, it suffices to show that $\lambda \geqslant 0$. This follows from three observations:

(a) Because $0 \leqslant w_\tau$, the function $\mathbb{E}_0^{\mathbb{Q}}[w_\tau((\bar{A}_\tau - w_\tau(1+\lambda))/2\xi)^+]$ is decreasing in λ.

(b) In Case 2, by hypothesis, we have

$$\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau}{2\xi}\right)^+\right] = \mathbb{E}_0^{\mathbb{Q}}[\mathbb{Q}_\tau w_\tau] > B.$$

(c) Finally, we know that λ solves

$$\mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right] = B.$$

Observations (a) and (b) therefore imply that we must have $\lambda \geqslant 0$. □

Case 1 of Proposition 6 describes the circumstances when trading in the financial market allows the retailer to completely remove the budget constraint from his optimization problem. When these circumstances are not satisfied as in Case 2, the retailer cannot completely remove the budget constraint. He can, however, mitigate the effects of the budget constraint somewhat so that for a fixed menu of wholesale prices, w_τ, he prefers the H-contract to the F-contract.

Based on the retailer's best-response strategy derived in Proposition 6, the producer's problem can be formulated as

$$\Pi_{\mathrm{P}}^{\mathrm{H}} = \max_{w_\tau,\, \lambda \geqslant 0} \ \mathbb{E}_0^{\mathbb{Q}}\left[(w_\tau - c_\tau)\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right] \tag{12}$$

$$\text{subject to } \mathbb{E}_0^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right] \leqslant B. \tag{13}$$

Note that at the optimal solution, the constraint in (13) will be tight if the optimal λ is greater than zero. This will occur only when the budget constraint is binding.

The following result characterizes the solution of this problem and the corresponding solution of the Stackelberg game.

Proposition 7 (Producer's Optimal Strategy and the Stackelberg Solution). *Let ϕ^{H} be the minimum $\phi \geqslant 1$ that solves*

$$\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2 - (\phi c_\tau)^2}{8\xi}\right)^+\right] \leqslant B.$$

Define $\delta^{\mathrm{H}} := \phi^{\mathrm{H}} c_\tau$; then, the optimal wholesale price and ordering level satisfy

$$w_\tau^{\mathrm{H}} = \frac{\bar{A}_\tau + \delta^{\mathrm{H}}}{2} \quad \textit{and} \quad q_\tau^{\mathrm{H}} = \left(\frac{\bar{A}_\tau - \delta^{\mathrm{H}}}{4\xi}\right)^+. \tag{14}$$

The players' expected payoffs satisfy

$$\Pi_{\mathrm{P}|\tau}^{\mathrm{H}} = \frac{(\bar{A}_\tau + \delta^{\mathrm{H}} - 2c_\tau)(\bar{A}_\tau - \delta^{\mathrm{H}})^+}{8\xi} \quad \textit{and}$$
$$\Pi_{\mathrm{R}|\tau}^{\mathrm{H}} = \frac{((\bar{A}_\tau - \delta^{\mathrm{H}})^+)^2}{16\xi}. \tag{15}$$

Proof. See the appendix.

As before, we interpret δ^{H} as a modified production cost, greater than or equal to the the original cost, c_τ, that is imposed in the supply chain because of the limited budget. Unlike the setting of the F-contract, however, the modified cost in this setting is not stochastic. Note that δ^{H} is nondecreasing in B. Hence, as in the F-contract, the more cash constrained the retailer, the higher the wholesale price charged by the producer.

Suppose now that the budget is limited so that $\delta^{\mathrm{H}} > c_\tau$. Then, depending on the value of δ^{H}, Proposition 7 implies that it is possible for $w_\tau^{\mathrm{H}} \geqslant \bar{A}_\tau$ and $q_\tau^{\mathrm{H}} = 0$ for some outcomes $\omega \in \Omega$. That is, in some cases the producer decides to overcharge the retailer and therefore make the supply chain nonoperative. Because of Assumption 1, this behavior was never optimal in the setting of the F-contract. It occurs in the H-contract setting, however, because the retailer can allocate his limited budget among different states $\omega \in \Omega$. In particular, if the retailer knows that for some outcomes, ω, he will not be purchasing any units, then he can transfer the entire budget B from these (nonoperative) states to states in which there is a need for cash. It is in the producer's interest, then, to select those states in which he wants to do business with the retailer and those in which he does not. Note that $q_\tau^{\mathrm{H}} = 0$ if and only if $\bar{A}_\tau \leqslant \delta^{\mathrm{H}}$. Hence, the producer "closes" the supply chain when the forecasted demand is low.

We now compare the F-contract with the H-contract in terms of the players expected payoffs under the Stackelberg equilibrium. First, we define

$$\widehat{\mathscr{X}} := \{\omega \in \Omega\colon\ \delta_\tau^{\mathrm{F}} = c_\tau\},$$

where δ_τ^{F} was defined in Proposition 1 in §3. The set $\widehat{\mathscr{X}}$ characterizes those states, ω, for which the flexible contract achieves the unconstrained optimal solution, $w_\tau^{\mathrm{F}} = (\bar{A}_\tau + c_\tau)/2$ and $q_\tau^{\mathrm{F}} = (\bar{A}_\tau - c_\tau)/4\xi$. We also recall that the equilibrium wholesale prices and ordering levels for the F-contract and H-contract are

$$w_\tau^{\mathrm{F}} = \frac{\bar{A}_\tau + \delta_\tau^{\mathrm{F}}}{2}, \quad q_\tau^{\mathrm{F}} = \frac{\bar{A}_\tau - \delta_\tau^{\mathrm{F}}}{4\xi} \quad \text{and}$$
$$w_\tau^{\mathrm{H}} = \frac{\bar{A}_\tau + \delta^{\mathrm{H}}}{2}, \quad q_\tau^{\mathrm{H}} = \frac{(\bar{A}_\tau - \delta^{\mathrm{H}})^+}{4\xi},$$

respectively. The difference between the expected payoffs of these two contracts depends on the difference between δ_τ^{F} and δ^{H}, which in turn depends on the set $\widehat{\mathscr{X}}$. We now show that the producer's expected payoff under the H-contract is always greater than his expected payoff under the F-contract.

Proposition 8. *The producer is always better off if the retailer is able to hedge the budget constraint.*

PROOF. See the appendix.

According to this result, it is in the producer's interest to promote the retailer's ability to trade in the financial market. If the retailer is a small player with limited access to the financial markets, then it would be in the producer's interest to serve as an intermediary between the retailer and the financial markets.

From the retailer's perspective, the comparison between the F-contract and the H-contract is not so straightforward. We identify three cases.

• *Case* 1. Suppose that $\widehat{\mathscr{X}} = \Omega$. In this case, B is sufficiently large so that $\delta_\tau^{\mathrm{F}} = \delta^{\mathrm{H}} = c_\tau$ for all $\omega \in \Omega$ and the two contracts produce the same output. This is not surprising because for large budgets, financial trading does not offer any advantage.

• *Case* 2. Suppose that $\widehat{\mathscr{X}} \neq \Omega$ and $\delta^{\mathrm{H}} = c_\tau$. In this case, $\delta_\tau^{\mathrm{F}} > c_\tau$ for all $\omega \in \widehat{\mathscr{X}}^c$. Therefore, $w_\tau^{\mathrm{H}} \leqslant w_\tau^{\mathrm{F}}$ and $q_\tau^{\mathrm{H}} \geqslant q_\tau^{\mathrm{F}}$ for all $\omega \in \Omega$ with strict inequalities in $\widehat{\mathscr{X}}^c$. With regard to the payoffs, using Equations (7) and (15) we can conclude that for all $\omega \in \Omega$,

$$\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}} \geqslant \Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}},$$

with strict inequality in $\widehat{\mathscr{X}}^c$. Note that this case summarizes well the advantages of using financial trading: the ability to trade has increased the output of the supply chain, reduced the wholesale price, reduced the double marginalization inefficiency, and increased the payoff of both agents. These conclusions hold for all $\omega \in \Omega$ in this case. Therefore, they hold in expectation, so that $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}}] \geqslant \mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}}]$.

• *Case* 3. Suppose that $\widehat{\mathscr{X}} \neq \Omega$ and $\delta^{\mathrm{H}} > c_\tau$. In this case, $\delta_\tau^{\mathrm{F}} < \delta^{\mathrm{H}}$ for $\omega \in \widehat{\mathscr{X}}$ and the wholesale price (ordering quantity) is smaller (higher) under the F-contract than under the H-contract. In terms of payoffs, the retailer (and the producer as well) therefore prefers the F-contract to the H-contract for $\omega \in \widehat{\mathscr{X}}$. Of course, the choice of the contract has to be made at $t = 0$ when the realization of ω is still unknown. Therefore, the appropriate comparison between the contracts should be based on their time $t = 0$ expected payoffs. As the following example shows, however, the retailer can be better off or worse off under the H-contract.

EXAMPLE 1. Consider the special case in which $\bar{A}_\tau$ takes only the values $\{5, 10\}$ with equal probability and $8\xi = 1$ and $B = 9.5$.

If $c_\tau = 1$, then we can show that $\delta^{\mathrm{H}} = 9 > c_\tau$ and $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}}] = 0.25$ and $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}}] = 0.342$.

If $c_\tau = 4.5$, then $\delta^{\mathrm{H}} = 9 > c_\tau$ and $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}}] = 0.25$ and $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}}] = 0.122$.

Example 1 shows that in some cases, the retailer can be worse off when he uses the financial market to hedge his budget constraint. In this case, the manufacturer is able to set a wholesale contract, w_τ^{H}, that exploits the retailer's ability to trade in the financial market to extract more of his operating profits. Hence, even if the retailer has access to the financial market, he might want the manufacturer to be unaware of this fact to induce the latter to offer the F-contract w_τ^{F} instead of the H-contract w_τ^{H}. Of course, this strategy can work only if the manufacturer is unable to observe whether or not the retailer is able to trade in the financial market. With asymmetric information, the manufacturer would have to set a different wholesale contract w_τ to handle this adverse selection problem.

When the budget B is sufficiently large (see Cases 1 and 2 above), the retailer is always better off using the H-contract. It turns out that under some additional conditions, we can show that for sufficiently small B the retailer is also better off under the H-contract. Therefore, it is only for intermediate value of B that the retailer might prefer the F-contract over the H-contract.

PROPOSITION 9. *Suppose that the random variable $\bar{A}_\tau$ has a bounded support and admits a smooth density bounded away from zero. Furthermore, assume that $\bar{A}_\tau > c_\tau$ for all $\omega \in \Omega$. Then, as $B \downarrow 0$, we obtain*

$$\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}}] = \xi B^2 \mathbb{E}_0^{\mathbb{Q}}\left[\frac{1}{\bar{A}_\tau^2}\right] + O(B^3) \quad \textit{and} \quad \mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}}] \geqslant KB^{3/2}$$

for some constant $K > 0$. Hence, for B sufficiently small, $\mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{F}}] \leqslant \mathbb{E}_0^{\mathbb{Q}}[\Pi_{\mathrm{R}\,|\,\tau}^{\mathrm{H}}]$.

PROOF. See the appendix.

According to the previous discussion, if $\delta^{\mathrm{H}} = c_\tau$, then both players are better off using the H-contract, so it follows that the entire supply chain is also better off. For the case $\delta^{\mathrm{H}} > c_\tau$, it is possible that the retailer prefers the F-contract and so it is not clear which contract has a higher total expected payoff, i.e., the sum of the retailer's and producer's expected profits.

Figure 3 shows the performance of the F-contract and the H-contract in terms of expected wholesale price, ordering level, and players' payoffs, as a function of the budget, B. It might be seen that if the budget is small, then on average, the wholesale price is smaller and the ordering level is higher for the F-contract than for the H-contract. This situation is reversed as the budget increases. In terms of the payoffs, both agents prefer the H-contract to the F-contract for all levels of B in this particular example. Furthermore, the benefits of the H-contract with respect to the F-contract are most pronounced for intermediate values of B.

4.2. Centralized Solution

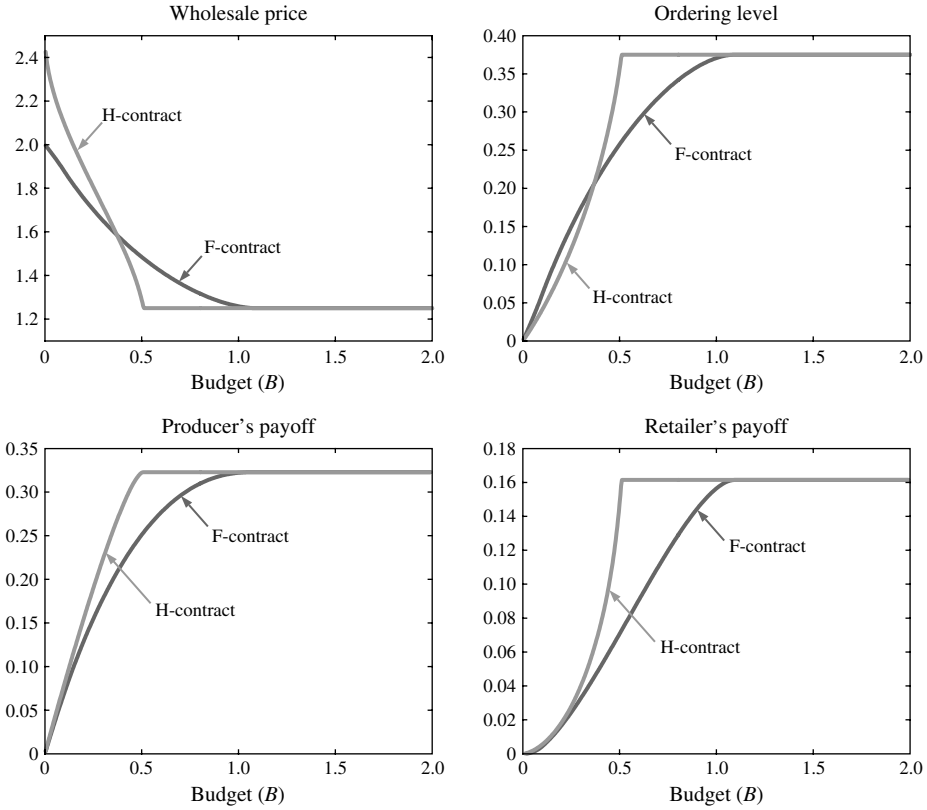
We now solve the centralized solution when the central planner is permitted to hedge the budget constraint. The central planner's problem is similar to the retailer's problem in (9)–(11). The only difference is that in this centralized operation, the procurement cost is c_τ instead of w_τ:

$$\Pi_{\mathrm{C}}^{\mathrm{H}} = \max_{q_\tau,\, B_\tau} \ \mathbb{E}_0^{\mathbb{Q}}[(\bar{A}_\tau - \xi q_\tau - c_\tau) q_\tau] \tag{16}$$

$$\text{subject to} \quad c_\tau q_\tau \leqslant B_\tau \quad \text{for all } \omega \in \Omega, \tag{17}$$

$$\mathbb{E}_0^{\mathbb{Q}}[B_\tau] = B. \tag{18}$$

Figure 3. Performance of the F-contract and the H-contract as a function of the budget.



Note. The demand parameter $\bar{A}_\tau$ is uniformly distributed in $[1, 3]$, $\xi = 1$, and $c_\tau = 0.5$.

Note that we have assumed that the central planner's budget is the same as the one that the retailer has in the decentralized solution. We make this assumption to compare the efficiency of the centralized and decentralized solutions without introducing any asymmetry on the financing of these two modes of operations.

Proposition 10 summarizes the optimal solution for the central planner. The proof is almost identical to the proof of Proposition 6 and is therefore omitted.

Proposition 10 (Central Planner's Optimal Strategy). *The optimal production strategy, $q^{\mathrm{H}}_{\mathrm{C}|\tau}$, is given by*

$$q^{\mathrm{H}}_{\mathrm{C}|\tau} = \left(\frac{\bar{A}_\tau - \delta^{\mathrm{H}}_{\mathrm{C}}}{2\xi}\right)^+, \tag{19}$$

where $\delta^{\mathrm{H}}_{\mathrm{C}}$ is the minimum $\delta \geqslant c_\tau$ that solves

$$\mathbb{E}^{\mathbb{Q}}\left[c_\tau\left(\frac{\bar{A}_\tau - \delta^{\mathrm{H}}_{\mathrm{C}}}{2\xi}\right)^+\right] \leqslant B.$$

The central planner's optimal payoff given the information available at time τ is

$$\Pi^{\mathrm{H}}_{\mathrm{C}|\tau} = \frac{(\bar{A}_\tau + \delta^{\mathrm{H}}_{\mathrm{C}} - 2c_\tau)(\bar{A}_\tau - \delta^{\mathrm{H}}_{\mathrm{C}})^+}{4\xi}. \tag{20}$$

Once again, we interpret $\delta^{\mathrm{H}}_{\mathrm{C}}$ as a modified production cost induced by the budget constraint.

4.3. Efficiency of the Centralized Solution

With this modified production cost structure in mind, one would expect the centralized solution to be more efficient than the decentralized solution in the sense that $\delta^{\mathrm{H}}_{\mathrm{C}} \leqslant \delta^{\mathrm{H}}$. This is not always the case, however, as the following example demonstrates.

Example 2. Consider the following instance of the problem with $B = 0.45$, $\xi = c_\tau = 1$, and $\bar{A}_\tau$ uniformly distributed in $[1, 3]$. Because

$$\mathbb{E}^{\mathbb{Q}}_0\left[\left(\frac{\bar{A}^2_\tau - c^2_\tau}{8\xi}\right)^+\right] = \frac{5}{12} < B \quad \text{and}$$

$$\mathbb{E}^{\mathbb{Q}}_0\left[c_\tau\left(\frac{\bar{A}_\tau - c_\tau}{2\xi}\right)\right] = \frac{1}{2} > B,$$

it follows that $c_\tau = \delta^{\mathrm{H}} < \delta^{\mathrm{H}}_{\mathrm{C}}$. Furthermore, we can shown that $\delta^{\mathrm{H}}_{\mathrm{C}} \approx 1.103$. Therefore, for values of $\bar{A}_\tau$ in $[1, \delta^{\mathrm{H}}_{\mathrm{C}})$, the central planner does not produce, i.e. $q^{\mathrm{H}}_{\mathrm{C}|\tau} = 0$, while the decentralized supply chain does operate, i.e. $q^{\mathrm{H}}_\tau > 0$. Because

$$\mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_\tau] = \mathbb{E}^{\mathbb{Q}}_0\left[\frac{\bar{A}_\tau - c_\tau}{4\xi}\right] = \frac{1}{4} \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_{\mathrm{C}|\tau}] = \frac{B}{c_\tau} = 0.45,$$

the central planner, on average, produces more than the decentralized supply chain.

The previous example highlights an interesting feature of the H-contract: contingent on the outcome ω, the centralized supply chain can produce less than the decentralized solution. This was never the case under the F-contract (or S-contract). As was the case with the manufacturer in the decentralized solution, the central planner uses the financial market to adjust his budget constraint, shutting down the operation (i.e., $q^{\mathrm{H}}_{\mathrm{C}|\tau}=0$) for some states ω of low demand and redistributing the cash that was originally spent in those states to states ω of high demand. Example 2 (above) shows that in some cases, when $\delta^{\mathrm{H}}<\delta^{\mathrm{H}}_{\mathrm{C}}$, the central planner chooses to close the supply chain for more states ω than the manufacturer in a decentralized operation. On average, however, the central planner always produces more than the decentralized supply chain. To see this, first note that if $\delta^{\mathrm{H}}_{\mathrm{C}}=c_\tau$, then (14) and (19) imply that $q^{\mathrm{H}}_{\mathrm{C}|\tau}\geqslant q^{\mathrm{H}}_{\tau}$, for all ω. However, if $\delta^{\mathrm{H}}_{\mathrm{C}}>c_\tau$, then Proposition 10 implies that $c_\tau\mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_{\mathrm{C}|\tau}]=B$. Then, Proposition 7, together with Assumption 1, imply that

$$B\geqslant \mathbb{E}^{\mathbb{Q}}_0[w^{\mathrm{H}}_\tau q^{\mathrm{H}}_\tau]=\mathbb{E}^{\mathbb{Q}}_0\left[\left(\frac{\bar{A}_\tau+\delta^{\mathrm{H}}}{2}\right)\left(\frac{\bar{A}_\tau-\delta^{\mathrm{H}}}{4\xi}\right)^+\right]$$
$$\geqslant c_\tau\mathbb{E}^{\mathbb{Q}}_0\left[\left(\frac{\bar{A}_\tau-\delta^{\mathrm{H}}}{4\xi}\right)^+\right]=c_\tau\mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_\tau],$$

implying, in particular, that $\mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_{\mathrm{C}|\tau}]\geqslant\mathbb{E}^{\mathbb{Q}}_0[q^{\mathrm{H}}_\tau]$. In terms of payoffs, it is clear that the central planner will always prefer the H-contract to the F-contract because the ability to hedge the budget constraint increases the set of feasible ordering quantities.

We conclude this section by examining the efficiency of the H-contract in terms of production levels, double marginalization, and the competition penalty. Toward this end, we define the following performance measures that are conditional on the information available at time τ:

$$\mathscr{Q}^{\mathrm{H}}_\tau:=\frac{q^{\mathrm{H}}_\tau}{q^{\mathrm{H}}_{\mathrm{C}|\tau}}=\frac{(\bar{A}_\tau-\delta^{\mathrm{H}})^+}{2(\bar{A}_\tau-\delta^{\mathrm{H}}_{\mathrm{C}})^+},\quad \mathscr{W}^{\mathrm{H}}_\tau:=\frac{w^{\mathrm{H}}_\tau}{c_\tau}=\frac{\bar{A}_\tau+\delta^{\mathrm{H}}}{2c_\tau},\quad\text{and}$$

$$\mathscr{P}^{\mathrm{H}}_\tau:=1-\frac{\mathbb{E}^{\mathbb{Q}}_0[\Pi^{\mathrm{H}}_{\mathrm{P}|\tau}]+\mathbb{E}^{\mathbb{Q}}_0[\Pi^{\mathrm{H}}_{\mathrm{R}|\tau}]}{\mathbb{E}^{\mathbb{Q}}_0[\Pi^{\mathrm{H}}_{\mathrm{C}|\tau}]}$$
$$=1-\frac{(3\bar{A}_\tau+\delta^{\mathrm{H}}-4c_\tau)(\bar{A}_\tau-\delta^{\mathrm{H}})^+}{4(\bar{A}_\tau+\delta^{\mathrm{H}}_{\mathrm{C}}-2c_\tau)(\bar{A}_\tau-\delta^{\mathrm{H}}_{\mathrm{C}})^+}.$$

It is interesting to note that, conditional on $\mathscr{F}_\tau$, the centralized supply chain is not necessarily more efficient than the decentralized operation. For example, we know that in some cases $\delta^{\mathrm{H}}<\delta^{\mathrm{H}}_{\mathrm{C}}$ (as in Example 2 above), so for all those ω with $\delta^{\mathrm{H}}<\bar{A}_\tau<\delta^{\mathrm{H}}_{\mathrm{C}}$, $q^{\mathrm{H}}_{\mathrm{C}|\tau}=0$ and $q^{\mathrm{H}}_\tau>0$, the competition penalty is arbitrarily negative. This never occurs under the F-contract. If $\delta^{\mathrm{H}}\geqslant\delta^{\mathrm{H}}_{\mathrm{C}}$, however, then it is easy to see that the centralized solution is always more efficient than the decentralized supply chain, so $\mathscr{Q}^{\mathrm{H}}_\tau\leqslant 1$ and $\mathscr{P}^{\mathrm{H}}_\tau\geqslant 0$.

We also note that if the budget is large enough so that both the decentralized and centralized operations can hedge away the budget constraint, then $\delta^{\mathrm{H}}=\delta^{\mathrm{H}}_{\mathrm{C}}=c_\tau$ and

$$\mathscr{Q}^{\mathrm{H}}_\tau=1/2\quad\text{and}\quad\mathscr{P}^{\mathrm{H}}_\tau=1/4.$$

5. Conclusions and Further Research

In this paper, we have studied the performance of a stylized supply chain where two firms, a retailer and a producer, compete in a Stackelberg game. The retailer purchases a single product from the manufacturer and then sells it in the retail market at a stochastic clearance price. The retailer, however, is budget constrained and is therefore limited in the number of units that he may purchase from the producer. We consider three types of contracts that govern the operation of the supply chain. In the case of the simple and flexible contracts, the retailer does not have access to the financial markets. In the case of the flexible contract with hedging, however, the retailer does have access to the financial markets, so he can, at least in part, mitigate the effects of the budget constraint. For each contract, we compare the decentralized competitive solution with the solution obtained by a central planner. We also compare the supply chain's performance across the different contracts.

Our model and results extend the existing literature on supply chain contracts by considering a budget-constrained retailer and by including financial markets as (i) a source of public information upon which procurement contracts can be written, and (ii) a means for financial hedging to mitigate the effects of the budget constraint.

We find that, in general, the more cash-constrained the retailer is, the higher the wholesale price charged by the producer. We also find that the producer always prefers the flexible contract with hedging to the flexible contract without hedging. Depending on the model parameters, however, the retailer might or might not prefer the flexible contract with hedging. One of our main results corresponds to Case 1 in Proposition 6. There we establish that if the budget is large enough, in an average sense, the ability to trade in the financial market allows the retailer to completely remove the budget constraint. This is not possible without financial trading unless the initial budget is so large that, regardless of the demand forecast $\bar{A}_\tau$, the budget constraint is never binding.

Another interesting feature of the solution of the H-contract is that when the forecasted demand is low (i.e., $\bar{A}_\tau$ is small), the producer chooses to shut down the supply chain by overcharging the retailer. By doing this, the producer can induce the retailer to transfer its limited budget from low-demand states to more profitable high-demand states. This is possible only if the retailer has access to the financial market to hedge the budget constraint. Under the F-contract, when access to the financial markets is not available, the producer never chooses to shut down the supply chain operations.

There are many directions in which this research could be extended. Caldentey and Haugh (2006b) consider the problem of choosing the optimal timing, τ, of the contract, and formulate this problem as an optimal stopping problem. They also extend the model of this paper to applications that incorporate foreign exchange risk, interest rate risk, and credit risk, among others.

There are many other directions future research could follow. First, it would be interesting to consider models where the nonfinancial noise evolved as an observable stochastic process. In this case, it would no longer be necessary for the trading gain, G_τ, to be $\mathcal{F}_\tau$-measurable. Indeed, the trading strategy would now depend on the evolution of both the financial and nonfinancial noise. Solving for the optimal trading strategy is then an incomplete-markets problem and would require mathematical techniques that are still being developed in the mathematical finance literature. Applying these techniques to our competitive Stackelberg-equilibrium setting where a budget constraint induces the desire to hedge would be particularly interesting and challenging.

A related direction for future research is to build and solve models where the need for hedging is induced by the presence of risk-averse agent(s) rather than the presence of a budget constraint. Caldentey and Haugh (2005, 2006a) consider such problems in a noncompetitive setting, where risk aversion is modeled by imposing explicit risk-management constraints or by assuming that the agent is risk averse with a quadratic utility function.

Third, it would be interesting to explore principal-agent problems in the setting where the risk-averse (or budget-constrained) agent has access to financial markets and the principal has imperfect information regarding the actions taken by the agent. Because the agent could use the financial market to smooth his income, it would presumably cost the principal agent less to ensure that the agent behaved optimally. This problem is, of course, related to the literature regarding executive compensation in corporate finance. In this literature, it is often the case that the agent or executive is *not* permitted to trade in his company's stock. However, there is no reason why the agent should not be free to trade in other financial markets that impact his company's performance. There are clearly many variations on this problem that could be explored.

A fourth direction would be to consider other types of contracts that the producer could offer to the retailer. In this paper, we have considered only linear price contracts, but other contracts could also be used. They include, for example, quantity discount, buy-back, and quantity flexibility contracts (e.g., Pasternack 1985 and Lovejoy 1999). A contract that might be of particular interest in our hedging framework is an *affine contract*, where the producer offers a contract of the form (w_τ, v_τ) to the retailer. In response, the retailer (assuming he accepts the contract) orders the random quantity, q_τ, and pays the producer $q_\tau w_\tau - v_\tau$, where v_τ is an $\mathcal{F}_\tau$-measurable random variable. If the retailer cannot trade, then this contract is very similar to our H-contract, where v_τ might be interpreted as a trading gain that is chosen by the producer. Obviously, this would result in an equilibrium that would differ from the equilibrium of the H-contract, where it is the retailer who chooses the trading gain. If the retailer did have access to the financial market, however, then this affine contract could be replaced by a contract of the form (w_τ, V), where V is now a constant transfer payment. This follows because the retailer could use the financial markets to capitalize the random gain, v_τ, obtaining instead $V := \mathbb{E}_0^{\mathbb{Q}}[v_\tau]$.

A particularly important direction for future research is to calibrate these models and operations financial market models more generally. This is not an easy task, but it will be necessary to do so if any of these models (competitive or noncompetitive) are to be implemented in practice. Accurate calibration would also enable us to determine what types of financial risks are worth hedging and what the resulting economic savings would be.

Appendix. Proofs of Propositions

Proof of Proposition 2. When $B \leqslant B_\tau^{\mathrm{F}}$ for all $\omega \in \Omega$, the inequalities follow from Jensen's inequality, the concavity of the function $f(x) = x + \sqrt{x^2 - 8\xi B}$, and the convexity of the functions $g(x) = x - \sqrt{x^2 - 8\xi B}$ and $h(x) = (x - \sqrt{x^2 - 8\xi B})^2$ in the region $x \geqslant 8\xi B$.

Let us now look at the retailer's payoff ratio when $B \downarrow 0$:

$$\lim_{B\downarrow 0} \frac{\Pi_{\mathrm{R}}^{\mathrm{F}}}{\Pi_{\mathrm{R}}^{\mathrm{S}}} = \lim_{B\downarrow 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau - \sqrt{\bar{A}_\tau^2 - 8\xi B}}{\bar{A} - \sqrt{\bar{A}^2 - 8\xi B}}\right)^2\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\lim_{B\downarrow 0}\left(\frac{\bar{A}_\tau - \sqrt{\bar{A}_\tau^2 - 8\xi B}}{\bar{A} - \sqrt{\bar{A}^2 - 8\xi B}}\right)^2\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\bar{A}}{\bar{A}_\tau}\right].$$

The second equality follows from the bounded convergence theorem and the third equality uses L'Hôpital's rule. A similar approach can be used to compute the limiting value of the producer's payoff ratio.

For the case $B \geqslant B_\tau^{\mathrm{F}}$ for all $\omega \in \Omega$, the equalities for w_τ^{F} and q_τ^{F} are straightforward. To verify the inequalities for the producer and retailer's payoff, note that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau - c)^2] &= (\bar{A} - c_0)^2 + \mathbb{E}^{\mathbb{Q}}[\bar{A}_\tau^2] - A^2 + 2\bar{A}c_0 \\ &\quad - 2\mathbb{E}^{\mathbb{Q}}[\bar{A}_\tau]c + c^2 - c_0^2 \\ &= (\bar{A} - c_0)^2 + \mathrm{Var}(\bar{A}_\tau) - 2\bar{A}(c - c_0) + c^2 - c_0^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\Pi_{\mathrm{M}}^{\mathrm{F}} \geqslant \Pi_{\mathrm{M}}^{\mathrm{S}} &\iff \mathbb{E}^{\mathbb{Q}}\left[\frac{(\bar{A}_\tau - c)^2}{8\xi}\right] \geqslant \frac{(\bar{A} - c_0)^2}{8\xi} \\ &\iff \mathrm{Var}(\bar{A}_\tau) + c^2 - c_0^2 \geqslant 2\bar{A}(c - c_0).\end{aligned}$$

The proof for the retailer's payoff is similar. □

Proof of Proposition 7. We prove a slightly more general result in which τ is a stopping time. The result in Proposition 7 is a special case for a deterministic time τ.

Consider an arbitrary $\mathcal{F}_t$-stopping time $\tau \leqslant T$ and the producer's optimization problem

$$\Pi_{\mathrm{M}}^{\mathrm{H}} = \max_{w_\tau,\, \lambda \geqslant 0} \mathbb{E}^{\mathbb{Q}}\left[(w_\tau - c_\tau)\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right]$$

$$\text{subject to } \mathbb{E}^{\mathbb{Q}}\left[w_\tau\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right] \leqslant B.$$

To solve this problem, we first relax the budget constraint using a multiplier $\beta \geqslant 0$. After relaxing the constraint, the new objective function becomes

$$\mathscr{L}(w_\tau, \lambda, \beta) := \mathbb{E}^{\mathbb{Q}}\left[(w_\tau(1-\beta)-c_\tau)\left(\frac{\bar{A}_\tau - w_\tau(1+\lambda)}{2\xi}\right)^+\right],$$

and it is clear that the optimal value of β will satisfy $\beta \leqslant 1$. In particular, we can restrict $\beta \in [0,1]$. We introduce the following change of variables:

$$y_\tau := w_\tau(1+\lambda) \quad \text{and} \quad \phi := \frac{1+\lambda}{1-\beta}.$$

Note that $\phi \geqslant 1+\lambda$ because $\beta \in [0,1]$ and $\lambda \geqslant 0$. We can now rewrite the objective function as

$$\mathscr{L}(y_\tau, \phi) = \frac{1}{\phi}\mathbb{E}^{\mathbb{Q}}\left[(y_\tau - c_\tau\phi)\left(\frac{\bar{A}_\tau - y_\tau}{2\xi}\right)^+\right].$$

Let us fix ϕ and optimize $\mathscr{L}(y_\tau, \phi)$ over y_τ. That is, we maximize $\mathscr{L}(y_\tau, \delta)$ pointwise for each y_τ. If $\bar{A}_\tau \geqslant c_\tau\phi$, then $y_\tau = (\bar{A}_\tau + c\phi)/2$ is optimal. If $\bar{A}_\tau \leqslant c_\tau\phi$, then any $y_\tau \geqslant \bar{A}_\tau$ is optimal. In particular, we can again take $y_\tau = (\bar{A}_\tau + c_\tau\phi)/2$ as the optimal solution. The corresponding optimal ordering quantity is given by

$$q_\tau = \left(\frac{\bar{A}_\tau - c_\tau\phi}{4\xi}\right)^+.$$

Now it remains only to find the optimal values of ϕ and λ. Given the previous solution, the producer's problem might be formulated as

$$\max_{\lambda\geqslant 0,\, \phi\geqslant 1+\lambda} \ \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2 - (c_\tau\phi)^2}{8\xi(1+\lambda)}\right)^+ - c_\tau\left(\frac{\bar{A}_\tau - c_\tau\phi}{4\xi}\right)^+\right]$$
$$\text{subject to} \ \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2 - c_\tau^2\phi^2}{8\xi(1+\lambda)}\right)^+\right] \leqslant B.$$

We can solve this problem as follows. Suppose that the optimal λ is strictly greater than zero. Then, the constraint must be binding because the objective function increases as λ decreases. But the first term in the objective function then equals B. Now note that it is possible to increase the objective function by increasing ϕ and maintain the tightness of the constraint by simultaneously reducing λ. (It is possible to do this because by assumption $\lambda > 0$.) Clearly, then we can continue increasing the objective function until $\lambda = 0$. In particular, we can conclude that the optimal value of λ is zero. The optimization problem may be now formulated as

$$\max_{\phi\geqslant 1} \ \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau - c_\tau\phi}{4\xi}\right)^+\left(\frac{\bar{A}_\tau + c_\tau\delta}{2} - c_\tau\right)\right]$$
$$\text{subject to} \ \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\bar{A}_\tau^2 - c_\tau^2\phi^2}{8\xi}\right)^+\right] \leqslant B.$$

By inspection, it is clear that the optimal solution, ϕ^*, satisfies $\phi^* = \max(1, \hat{\phi})$, where $\hat{\phi}$ is the value of ϕ that makes the constraint binding.

The statement of Proposition 7 is complete once we identify δ^{H} with $c_\tau\phi^*$. □

Proof of Proposition 8. Suppose first that $\delta^{\mathrm{H}} = c_\tau$. Under the F-contract, the manufacturer's expected payoff can be written as

$$\mathbb{E}^{\mathbb{Q}}[\Pi^{\mathrm{F}}_{\mathrm{M}|\mathrm{X}}] = \frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau + \delta^{\mathrm{F}}_\tau - 2c_\tau)(\bar{A}_\tau - \delta^{\mathrm{F}}_\tau)^+], \quad \text{where}$$
$$\delta^{\mathrm{F}}_\tau = \max\left\{c_\tau, \sqrt{(\bar{A}_\tau^2 - 8\xi B)^+}\right\}.$$

For $\delta^{\mathrm{H}} = c_\tau$, it is a matter of simple calculations to show that

$$\frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau + \delta^{\mathrm{F}}_\tau - 2c_\tau)(\bar{A}_\tau - \delta^{\mathrm{F}}_\tau)^+]$$
$$\leqslant \frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau + \delta^{\mathrm{H}} - 2c_\tau)(\bar{A}_\tau - \delta^{\mathrm{H}})^+] = \mathbb{E}^{\mathbb{Q}}[\Pi^{\mathrm{H}}_{\mathrm{M}|X}],$$

so the manufacturer's payoff is better under the H-contract than under the F-contract.

Suppose now that $\delta^{\mathrm{H}} > c_\tau$, and let us consider the following optimization problem:

$$\max_{\delta_X} \ \frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau + \delta_X - 2c_\tau)(\bar{A}_\tau - \delta_X)^+] \tag{21}$$
$$\text{subject to} \ \frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau^2 - \delta_X^2)^+] \leqslant B, \tag{22}$$
$$\delta_X \geqslant c_\tau \quad \text{for all } X \in \mathscr{X}. \tag{23}$$

Note that $\{\delta^{\mathrm{F}}_\tau\colon X \in \mathscr{X}\}$ is a feasible solution for this problem. Hence, to complete the proof of the proposition it is enough to show that $\delta^*_X = \delta^{\mathrm{H}}$ for all $X \in \mathscr{X}$ is an optimal solution to (21)–(23). To prove this, first note that because $\delta^{\mathrm{H}} > c_\tau$, constraint (22) must be bidding at optimality. This follows from the fact that $(\bar{A}_\tau + \delta_X - 2c_\tau)(\bar{A}_\tau - \delta_X)^+$ is decreasing in δ_X in the range $\delta_X \in [c_\tau, \bar{A}_\tau]$. Therefore, a solution to (21)–(23) also solves

$$\min_{\delta_X} \ \mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau - \delta_X)^+] \tag{24}$$
$$\text{subject to} \ \frac{1}{8\xi}\mathbb{E}^{\mathbb{Q}}[(\bar{A}_\tau^2 - \delta_X^2)^+] = B, \tag{25}$$
$$\delta_X \geqslant c_\tau \quad \text{for all } X \in \mathscr{X}. \tag{26}$$

To solve this problem, we relax constraint (25). The corresponding Lagrangian function is

$$\mathscr{L}(\delta, \lambda) \triangleq \mathbb{E}^{\mathbb{Q}}[\mathscr{L}_X(\delta_X, \lambda)], \quad \text{where}$$
$$\mathscr{L}_X(\delta_X, \lambda) \triangleq (\bar{A}_\tau - \delta_X)^+(1 + \lambda(\bar{A}_\tau + \delta_X)).$$

If $\lambda \geqslant 0$, then $\delta_X \geqslant \bar{A}_\tau$ for all X minimizes $\mathscr{L}(\delta, \lambda)$. However, this solution does not satisfy constraint (25). Hence,

we must have $\lambda < 0$. In this case, the problem of minimizing $\mathcal{L}(\delta, \lambda)$ is solved with $\delta_X = -1/2\lambda$ for all X, that is, a constant value. To pick this fix value of δ_X, we need to impose constraints (25) and (26). By the definition of δ^{H}, we conclude that $\delta_X = \delta^{\mathrm{H}}$ for all X. □

Proof of Proposition 9. Let $[A_l, A_u]$ and f_A be the support and density of $\bar{A}_\tau$, respectively. Note that by assumption $A_l > c_\tau$, and there is an $\epsilon > 0$ such that $f_A(z) > \epsilon$ for all $z \in [A_l, A_u]$.

For B sufficiently small, $\delta_\tau^{\mathrm{F}} = \sqrt{\bar{A}_\tau^2 - 8\xi B} = \bar{A}_\tau - 4\xi B/\bar{A}_\tau + O(B^2)$. Therefore, as $B \downarrow 0$, the retailer's payoff satisfies

$$\mathbb{E}^{\mathbb{Q}}[\Pi^{\mathrm{F}}_{\mathrm{R}|\mathrm{X}}] = \frac{1}{16\xi}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{4\xi B}{\bar{A}_\tau} + O(B^2)\right)^2\right] = \xi B^2 \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\bar{A}_\tau^2}\right] + O\big(B^3\big).$$

For the case of the H-contract, for B sufficiently small $\delta^{\mathrm{H}} > c_\tau$, and solves

$$\int_{\delta^{\mathrm{H}}}^{A_u}\left(\frac{z^2 - (\delta^{\mathrm{H}})^2}{8\xi}\right) f_A(z)\,\mathrm{d}z = B.$$

According to the mean-value theorem, there is an $\tilde{A} \in [\delta^{\mathrm{H}}, A_u]$ such that

$$f_A(\tilde{A})\int_{\delta^{\mathrm{H}}}^{A_u}\left(\frac{z^2 - (\delta^{\mathrm{H}})^2}{8\xi}\right)\mathrm{d}z = B.$$

After integrating and some straightforward manipulations, we get

$$(A_u - \delta^{\mathrm{H}})^2 = \frac{24\xi B}{(A_u + 2\delta^{\mathrm{H}}) f_A(\tilde{A})}. \tag{27}$$

The retailer's payoff under the H-contract satisfies

$$\mathbb{E}^{\mathbb{Q}}[\Pi^{\mathrm{H}}_{\mathrm{R}|\mathrm{X}}] = \frac{1}{16\xi}\int_{\delta^{\mathrm{H}}}^{A_u}(z - \delta^{\mathrm{H}})^2 f_A(z)\,\mathrm{d}z = \frac{f_A(\hat{A})}{16\xi}\int_{\delta^{\mathrm{H}}}^{A_u}(z - \delta^{\mathrm{H}})^2\,\mathrm{d}z = \frac{f_A(\hat{A})(A_u - \delta^{\mathrm{H}})^3}{48\xi}$$

for some $\hat{A} \in [\delta^{\mathrm{H}}, A_u]$. Hence, we can combine this identity and condition (27) to get

$$\mathbb{E}^{\mathbb{Q}}[\Pi^{\mathrm{H}}_{\mathrm{R}|\mathrm{X}}] = \frac{f_A(\hat{A})}{48\xi}\left(\frac{24\xi}{(A_u + 2\delta^{\mathrm{H}}) f_A(\tilde{A})}\right)^{3/2} B^{3/2} \geqslant K B^{3/2},$$

where the constant K satisfies

$$K = \sqrt{\frac{2\xi}{9A_u^3}}\min\left\{\frac{f_A(\hat{A})}{f_A(\tilde{A})^{3/2}}\colon \tilde{A}, \hat{A} \in [A_l, A_u]\right\} > 0.$$

The inequality follows from the fact that f_A is bounded away from zero. □

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