Production Postponement with Borrowing and Hedging in a Competitive Supply Chain

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Abstract

We investigate the impact of production postponement on the operations of a stylized supply chain where N identical retailers and a single producer compete in a Cournot-Stackelberg game. The retailers purchase a single product from the producer and afterwards sell it in the retail market at a stochastic clearance price. We assume the retailers' profits depends in part on the realized path or terminal value of some tradeable financial market such as a foreign exchange rate, commodity index or more generally, any relevant economic index. We therefore consider a variation of the traditional wholesale price contract that is offered by the producer to the retailers. Under this contract, at $t = 0$ the producer offers a menu of wholesale prices to the retailers, one for each realization of the financial process up to some future time τ. The retailers then commit to purchasing at time τ a variable number of units, with the specific quantity depending on the realization of the process up to time τ. We assume the retailers are budget constrained but are able to hedge and borrow in the financial markets to partly mitigate this. We completely characterize the resulting Cournot-Stackelberg equilibrium and compare it to various benchmarks including equilibria where hedging and / or borrowing are not available to the retailers. We show there is a pecking order to the hedging and debt components of the financial markets. Specifically, hedging is used at low and intermediate budget levels while debt is only used at low budget levels. We identify conditions under which the producer, retailers, consumers and a central planner are all better off by postponing production. We also study the impact of retail competition on the equilibrium. We show that higher levels of competition in the retailers' market increase supply chain efficiency, consumers' surplus and social welfare when the retailers' budgets are either high or low. For intermediate budget levels, however, it's possible that too much retailer competition can have a detrimental effect on these measures.

Subject Classifications: Finance: portfolio, management. Non-cooperative Games: applications. Production: applications.

Keywords: Production postponement, procurement contract, financial constraints, supply chain coordination.

1 Introduction

Production postponement – the delay of processing, distribution and other types of product differentiation activities – is arguably one of the fundamental pillars in the operations of modern supply chains and supports many quick-response, mass-customization and risk-management initiatives. Accordingly, there is a sizeable Operations Management literature investigating the benefits of production postponement and the challenges associated with its implementation (see Feitzinger and Lee, 1997, Cheng et al., 2010 and Section 1.1 for additional references). For the most part, this literature centers around operational issues in the design-make-ship cycle and ranges from modular product design to selecting optimal inventory localization strategies to building agile logistics networks. Our aim is to contribute to this literature by expanding the scope of production postponement in the context of the cash-flow management of financially constrained supply chains.

We investigate the impact of '*financial postponement*' in the operations of a stylized supply chain with the following distinctive characteristics. First, there are N retailers and a single producer competing in a Cournot-Stackelberg game to serve a future random demand. Second, retailers have a finite budget that limits the number of units they may purchase from the producer. Third, consumers' demand depends in part on the realized path or terminal value of some observable and tradeable economic index such as a foreign exchange rate, commodity price or more generally, some tradeable financial process. Finally, the producer and retailers negotiate the terms of a wholesale price contract with a production postponement provision under which retailers are able to delay their final procurement orders to a later time. We argue that the value of postponement is two-fold in this setting. First, there is the *informational* benefit of providing retailers additional time to improve their demand forecasts by tracking the evolution of the financial market. Second, postponement offers a *financial* benefit by giving the retailers the opportunity to trade dynamically in the financial market to reallocate their financial resources, i.e. their budgets, across different states of nature so as to make them contingent upon better demand forecasts. In other words, financial trading provides retailers a chance to hedge demand uncertainty by allowing them to allocate more budget to those states where demand is likely to be high and less budget to states where the demand is likely to be low. In addition to hedging their budget constraints, we also assume that retailers can borrow in the financial market in order to increase the amount of inventory they can procure. We assume that any such borrowing, however, is costly thereby reflecting market frictions such as transactions costs, liquidity costs, default risk, etc.

In this paper we will explore a number of questions that emerge from our supply chain model. At one end, there are issues related to how postponement impacts market efficiency by modulating the interplay between procurement decisions and financial strategies, i.e. hedging and borrowing. For example, is it always true that postponement leads to higher levels of output, better service levels for consumers, and higher profits for the retailers and the producer? As we shall see, the answers to these questions are not uniform and will depend on the level of competition in the retailers' market and on their initial budgets. Another set of questions focus on understanding how production postponement and financial trading impact the outcome of the Cournot-Stackelberg equilibrium between the producer and the retailers. For example, how should a strategic producer adjust the terms of an optimal wholesale price contract to extract some of the gains the retailers perceive from postponing their ordering decisions and from engaging in financial trading. Because orders are not placed until a later date, we will see that the producer effectively offers a random wholesale price which is contingent upon the evolution of the financial market. Alternatively, we can view this contract as a *menu* of wholesale prices that the producer offers the retailers at the time the contract is negotiated. Each price in the menu is then associated with one realization of the financial market.

In terms of our analysis and results, we completely characterize the resulting Cournot-Stackelberg equilibrium and compare it to various benchmarks where one or both of hedging and borrowing are not available to the retailers. We study the impact of the financial markets (as a source of information as well as a mechanism to hedge and borrow) on the various players including the firms themselves, the end consumers and society as a whole. We also study the impact of retail competition on the supply chain. In particular, we study the impact of varying N, i.e. the level of competition in the the retailers' market, on the Cournot-Stackelberg equilibrium whilst keeping the aggregate budget of all retailers fixed.

Summary of Key Takeaways: In contributing to the work in the interface between finance and operations management, our main contributions center around providing a better understanding of how production postponement combined with access to financial markets impact both the operations of a financially constrained supply chain and the welfare of consumers and society. We also provide a better understanding of how these effects vary with the degree of market competition in the retailers' market. Some key takeaways of our work are the following:

- There is a pecking order to the hedging and debt components of the financial markets in an optimal financial strategy. Specifically, hedging is used at all budget levels lower than some fixed threshold $\overline{\overline{B}}$. In contrast, (costly) debt is only used (in conjunction with hedging) when the retailers' budgets are smaller than another threshold level $\underline{B} < \overline{\overline{B}}$. These thresholds are determined endogenously as part of the equilibrium.

- By delaying the time at which final orders are placed, one expects that retailers would benefit from (i) better demand forecasts, (ii) the opportunity to hedge their budget constraints and optimize the allocation of their financial resources and (iii) reduce the cost of debt by borrowing only on those states where interest rates are the lowest. Surprisingly perhaps, despite all these benefits, it is possible that production postponement can hurt retailers.

- The producer can be significantly better off in equilibrium when he is free to offer a state dependent price menu as opposed to a constant price menu. In contrast, the retailers can be significantly worse off.

- Even when there are costs to delaying production, we identify sufficient conditions (when B is sufficiently large or in the limit as B goes to zero) under which all agents (producer, retailer, overall supply chain, consumers and social welfare) are better off when production is delayed.

- Higher levels of retailer competition do not necessarily lead to an increase in supply chain efficiency, consumer surplus or social welfare. These positive outcomes of retailer competition are possible only when the retailers' initial budgets are either high or low. For intermediate budget levels, however, it's possible that too much retail competition can have a detrimental effect on these measures. In particular there may exist a finite number of retailers N^* which is optimal from a consumer surplus and social welfare point of view.

1.1 Related Literature

There exists an extensive literature in operations devoted to production postponement. See for example Yang et al. (2004), Boone et al. (2006), Cheng et al. (2010) and the recent historical review by Zinn (2019). Within this literature, *delayed production differentiation* and *quick response* are arguably the two forms of postponement that have received most of the attention. In the former case, postponement is achieved by redesigning products and production processes to delay their point of differentiation, i.e., the stage in the process at which work-in-process is transformed into a unique finished product; see Lee and Tang (1997), Garg and Tang (1997), Swaminathan and Tayur (1998), Aviv and Ferdergruen (2001), Swaminathan and Lee (2003)). In contrast, quick response initiatives involve reducing supply chain lead times and other forms of distribution delays so that inventory localization decisions are postponed as much as possible with the objective of improving demand forecasts and minimizing logistics costs; see Fisher and Raman (1996), Iyer and Bergen (1997), Choi and Sethi (2010).

A distinguishing feature of our model with respect to most of the postponement literature in supply chain management is the budget constraint that we impose on the retailers' procurement decisions. When firms are financially constrained, production postponement offers an additional benefit which has received little attention in the literature. In particular, production postponement delays the time at which financial transactions and payments among firms are executed and therefore provides firms an opportunity to use financial markets and other sources of financing to mitigate the costs imposed by their limited budgets. Some papers that investigate related issues include Buzacott and Zhang (2004), Caldentey and Chen (2012), Caldentey and Haugh (2009), Dada and Hu (2008), Hu and Sobel (2005), Gupta and Chen (2016), Kouvelis and Zhao (2012), Kouvelis and Zhao (2016), and Xu and Birge (2004); see also Part Three in Kouvelis et al. (2012). These papers generally consider various mechanisms such as asset-backed financing or bank borrowing to mitigate the impact of the budget constraint.

The work by Caldentey and Haugh (2009), Caldentey and Chen (2012) and Kouvelis and Zhao (2012) are most closely related to this paper. They all consider a two-echelon supply chain system in which there is a single budget constrained retailer and they investigate different types of procurement contracts between the agents using a Stackelberg equilibrium concept. Caldentey and Chen (2012) discuss two alternative forms of financing for the retailer: (a) *internal financing* in which the supplier offers a procurement contract that allows the retailer to pay *in arrears* a fraction of the procurement cost after demand is realized and (b) *external financing* in which a third party financial institution offers a commercial loan to the retailer. They conclude that in an optimally designed contract it is in the supplier's best interest to offer financing to the retailer and that the retailer will always prefer internal rather than external financing. In a similar setting, Kouvelis and Zhao (2012) consider a supply chain in which the supplier offers different type of contracts designed to provide financial services to the retailer. They analyze a set of alternative financing schemes including *supplier early payment discount*, *open account financing*, *joint supplier financing with bank*, and *bank financing* schemes.

In Caldentey and Haugh (2009) the supplier offers a modified wholesale price contract to a single budget constrained retailer and the contract is executed at a future time τ. The terms of the contract are such that the actual wholesale price charged at time τ depends on information publicly available at this time. Delaying the execution of the contract is important because in this model the retailer's demand depends in part on a financial index that the retailer and supplier can observe through time. As a result, the retailer can dynamically trade in the financial market to adjust his budget to make it contingent upon the evolution of the index. Their model shows how financial markets can be used as (i) a source https://www.overleaf.com/project/614b4bcb7cec78cb69415f6bof public information upon which procurement contracts can be written and (ii) as a means for financial hedging to mitigate the effects of the budget constraint. In this paper, we therefore extend the model in Caldentey and Haugh (2009) by considering a market with *multiple retailers* in Cournot competition as well as a Stackelberg leader. One of the distinguishing features of having multiple retailers is that it allows us to study the impact of hedging upon competition in the retailers' market. We also extend Caldentey and Haugh (2009) by allowing the retailers to have access to costly borrowing which can complement the retailers' financial hedging activity. Allowing for the possibility of costly borrowing provides a more realistic representation of real-world corporate finance activity and as we shall see, also accounts for some interesting properties of the full Cournot-Stackelberg equilibrium. Finally, we note that Caldentey and Haugh (2017) state (without proof) some basic results for a model similar to the one we consider in this paper. They do not allow for the possibility of risky borrowing, however, and do not explore any of the issues related to production postponement and assessing the value of financial markets as we do here in this work.

There is a large literature on the use of debt financing in inventory and supply chain management and we certainly can't do justice to it here. In this paper we use costly debt financing as a potential substitute / complement to hedging but there are many other reasons for studying debt financing. To give just a few recent examples, Iancu et al. (2017) study the inefficiencies arising from a firm's operating flexibility under debt. They note that debt financing and operating flexibility could lead to borrowing costs that erase a significant amount of the firm's value and study the effectiveness of covenants in limiting this value destruction. In a game-theoretic setting with two firms they show that even firms with unlimited internal capital may prefer external debt financing in order to mitigate problems that are associated with knowledge spillover, e.g. free-riding. Finally, Besbes et al. (2018) study dynamic pricing when inventory is debt-financed. They note that pricing distortions can arise when there is limited liability under debt and that these distortions result in revenue losses that compound over time. They propose various partial remedies for these problems including the use of debt amortization and financial covenants.

Another related stream of research considers Cournot-Stackelberg equilibria. There is an extensive economics literature on this topic that focuses on issues of existence and uniqueness of the Nash equilibrium. See Okoguchi and Szidarovsky (1999) for a comprehensive review. In the context of supply chain management, there has been some recent research that investigates the design of efficient contracts between the supplier and the retailers. For example, Bernstein and Federgruen (2003) derive a perfect coordination mechanism between the supplier and the retailers. This mechanism takes the form of a nonlinear wholesale pricing scheme. Zhao et al. (2005) investigate inventory sharing mechanisms among competing dealers in a distribution network setting. Li (2002) studies a Cournot-Stackelberg model with asymmetric information in which the retailers are endowed with some private information about market demand. In contrast, the model we present in this paper uses the *public* information provided by the financial markets to improve the supply chain coordination.

There also exists a related stream of research that investigates the use of financial markets and instruments to hedge operational risk exposure. See Boyabatli and Toktay (2004) and the survey paper by Zhao and Huchzermeier (2015) for detailed reviews. For example, Caldentey and Haugh (2006) consider the general problem of dynamically hedging the profits of a risk-averse corporation when these profits are partially correlated with returns in the financial markets. Ding et al. (2007) and Dong et al. (2014) examine the interaction of operational and financial decisions from an integrated risk management standpoint. Boyabatli and Toktay (2011) analyze the effect of capital market imperfections on a firm's operational and financial decisions in a capacity investment setting. Wang and Yao (2017) consider the joint problem of optimizing over a one-time production quantity and an associated dynamic hedging strategy. With a mean-variance objective they succeed in completely characterizing the efficient frontier as well as the improvement in risk-return tradeoff that results from the hedging strategy. See also Wang and Yao (2016) for work in a similar vein where a shortfall objective is also considered.

Finally, there is an extensive stream of research in the corporate finance literature that relates to financial risk management and that is closely related to this paper. Of course the Modigliani-Miller theorem (Modigliani and Miller, 1958) states that firms, in the absence of market frictions, do not need to hedge since individual shareholders can do so themselves. In practice, however, there are many frictions that necessitate firm hedging and it is well known (see, for example, Boyle and Boyle, 2001) that many firms do so. These frictions include taxes and the costs of financial distress (Smith and Stulz, 1985), managerial motives (Stulz, 1984) as well as the costs associated with external financing (Stulz, 1990, Lessard, 1991). The work of Froot et al. (1993) was particularly influential and, building upon the earlier work of Lessard (1991), argues that the most important driver of firm hedging are the costs associated with external financing. In a two-period model they explicitly derive the optimal hedging strategy together with optimal financing and investing decisions for a single firm with costly external financing. This is very much in the spirit of our paper where the retailers are budget constrained and financial hedging allows them to mitigate the effect of these constraints. As with Froot et al. (1993), we also allow for the possibility of costly external financing but our set-up with multiple retailers and a producer in competition is more complex.

Adam et al. (2007) also assume a two-period model with firms that are identical ex-ante. They focus on determining what percentage of the firms will hedge in a Cournot equilibrium framework. In contrast to our work, it is therefore not the case that every firm will have an incentive to hedge in equilibrium. For tractability reasons, they also assume external financing is not possible. Other more recent papers also consider firm hedging in a game-theoretic framework. For example, Pelster (2015) considers hedging in a duopoly framework with mean-variance preferences while Loss (2012) also considers a duopoly and concludes that the firms' hedging demands decrease with the correlation between their internal funds and investment opportunities. Liu and Parlour (2009) consider a Cournot hedging framework where players can hedge the cash-flows from an indivisible project but not the probability of winning the project in an auction setting. In contrast to our work, none of these papers consider a Cournot-Stackelberg framework and they generally assume only very simple forms of hedging, e.g. via forward contracts, are allowed. Moreover, these papers take a more financial perspective and don't explicitly model the supply chain dynamics which is in contrast to our work here.

The remainder of this paper is organized as follows. In Section 2 we describe our model, specifically the supply chain, the financial markets and the contractual agreement between the producer and the retailers. In Section 3 we solve for the Cournot equilibrium where the producer's price menu is fixed and study some of its properties there. In Section 4 we completely characterize the full

Cournot-Stackelberg equilibrium where the producer optimizes over his price menu taking the best response of the retailers into account. In Section 5 we study the interplay between production postponement and access to the financial markets in terms of supply chain gains and social welfare. In this context we also study the performance of the full Cournot-Stackelberg with the equilibria of various benchmarks where access to the financial markets is either limited or not possible at all. In Section 6 we consider the effects of competition among retailers in the full equilibrium. In particular we study how the Cournot-Stackelberg equilibrium varies as a function N while keeping the aggregate retail budget fixed. The appendices contains proofs of technical results, equilibrium derivations for the various benchmarks as well as some further technical discussion.

2 Model Description

We now describe the model in further detail and we begin with a description of the supply chain including the firms' decisions, payoffs and the terms of the procurement contracts. We then discuss the role of the financial markets and formulate the Cournot-Stackelberg game between the producer and retailers. We conclude the section with a discussion of our modeling assumptions.

2.1 The Supply Chain

We model[1] an isolated segment of a competitive supply chain with one producer that produces a single product and N identical budget-constrained retailers that compete on the same retail market. The operating horizon $[0, T]$ is divided into three time epochs:

1. At time $t = 0$ the producer and retailers negotiate the terms of a procurement contract.
2. At time $t = \tau \geq 0$ the retailers place their procurement orders and pay the producer for these orders.
3. At time $t = T \geq \tau$ the producer fulfills the retailers' orders, the market uncertainty is resolved and the retailers sell the product in the retail market.

Procurement Contract: The first component of our model is the contractual agreement between the producer and the retailers. We consider a variation of the traditional wholesale price contract with a postponement provision in which the terms of the contract are specified contingent upon the information that is publicly available in the financial markets at the future time τ. Specifically, at time $t = 0$ the producer offers an $\mathcal{F}_\tau$-measurable wholesale price w_τ to the retailers, where $\mathcal{F}_\tau$ is a σ-algebra modeling time τ information available in the financial markets; see Section 2.2 below. In response to this offer, the i^{th} retailer decides on an $\mathcal{F}_\tau$-measurable ordering quantity[2] $q_{i\tau} = q_{i\tau}(w_\tau)$ for $i = 1, \ldots, N$. To be clear, the contract itself is negotiated at time $t = 0$ whereas the actual order quantities are only realized at time $\tau \geq 0$. However, an alternative and equivalent interpretation is that w_τ is announced at time $t = 0$ and the retailers do not place

[1] Similar models are discussed in detail in Section 2 of Cachon (2003). See also Lariviere and Porteus (2001).

[2] There is a slight abuse of notation here and throughout the paper when we write $q_i = q_i(w_\tau)$. This expression should not be interpreted as implying that $q_{i\tau}$ is a function of w_τ. We only require that q_i be $\mathcal{F}_\tau$-measurable and so a more appropriate interpretation is to say that $q_{i\tau} = q_{i\tau}(w_\tau)$ is the retailer's response to w_τ.

their procurement orders until time τ. It will often be convenient to take this latter interpretation. Either way, the retailers pay the producer for the units they have ordered at time τ. The producer fulfills the retailers' orders at time $T \geq \tau$ at which point the retailers sell the product units in the retail market and realize their profits (or losses).

We assume the producer offers the same contract to each retailer and that during the negotiation of the contract the producer acts as a Stackelberg leader. That is, the producer moves first and at $t = 0$ proposes a wholesale price menu w_τ to which the retailers then respond by selecting their ordering levels $q_{i\tau}$ for $i = 1, \ldots, N$. The N retailers also compete among themselves in a Cournot-style game to determine their optimal ordering quantities.

Retailer's Budget Constraint: We assume each retailer has an initial budget that may be used to purchase product units from the producer. In particular, we assume that retailer's are symmetric and each has the same initial budget B. In the absence of any financial strategy, the order quantity $q_{i\tau}$ must satisfy the budget constraint $w_\tau \, q_{i\tau} \leq B$ in each time τ state. In Section 2.2 we will discuss how the retailers can use the financial markets to relax their budget constraints.

Producer's Payoff: We assume the producer has unlimited production capacity with a per-unit production cost of c_τ. We will assume that c_τ is deterministic although many of our results go through when c_τ is stochastic. The producer's time τ payoff as a function of the wholesale price w_τ and the ordering quantities $q_{i\tau}$ is given by

$$\Pi_{\mathrm{P}|\tau} := (w_\tau - c_\tau) \sum_{i=1}^{N} q_{i\tau}. \tag{1}$$

Throughout the paper, the subscript τ is used to denote the value of a quantity conditional on time τ information. Finally, the producer's expected time 0 payoff is denoted by $\Pi_{\mathrm{P}} = \mathbb{E}[\Pi_{\mathrm{P}|\tau}]$.

Retailers' Payoffs: For a given set of order quantities $\{q_{i\tau} : i = 1, \ldots, N\}$, the i^{th} retailer collects a revenue net of procurement costs equal to $(P(Q_\tau) - w_\tau)\, q_{i\tau}$, where $P(Q_\tau)$ is the (random) *clearance price* in the retail market given a total inventory $Q_\tau := \sum_i q_{i\tau}$. We further assume that the clearance price admits the linear representation $P(Q) := A - Q$, where A is a non-negative random variable that models the market size (or market potential) of the product at time T. Also, since the retailers' procurement quantities are $\mathcal{F}_\tau$-measurable, it will be convenient to define the expectation of A conditional on $\mathcal{F}_\tau$ and we denote this by $A_\tau := \mathbb{E}_\tau[A]$. The dependence between the market size and the financial markets is therefore captured via A_τ. For example, if A was independent of the financial markets then A_τ would be a constant and so the optimal ordering quantities and price menu, being functions of A_τ (see Section 3), would also be constant.

The expected payoff of the i^{th} retailer conditional on time τ information therefore takes the form

$$\Pi_{\mathrm{R}_i|\tau} \;:=\; \left(A_\tau - \left(q_{i\tau} + Q_{i\tau-}\right)\right) q_{i\tau} - w_\tau \, q_{i\tau} \tag{2}$$

where $Q_{i\tau-} := \sum_{j \neq i} q_{j\tau}$ is the cumulative inventory position of all the retailers excluding retailer i. Note that $\Pi_{\mathrm{R}_i|\tau}$ is a function of w_τ and the vector of order quantities. Finally, retailer i's expected payoff at time 0 is denoted by $\Pi_{\mathrm{R}_i} := \mathbb{E}[\Pi_{\mathrm{R}_i|\tau}]$.

Remark 1 (Clearance Price) *It is customary in the inventory and supply chain management literature to model market uncertainty in the form of a random consumer's demand. In most practical* *situations, however, excess inventory and unsold units are generally liquidated using secondary markets at discount prices. Hence, we interpret our stochastic clearance price as the appropriately weighted average selling price across all units and markets. In addition, we note that the linear clearance price model is commonly assumed in the operations and economics literature for reasons of tractability and estimation. It also helps ensure that the Cournot game among retailers will have a unique Nash equilibrium.*

2.2 The Financial Market

We assume that each of the retailers has access to the financial markets where they can engage in two types of financial activities which we refer to as hedging and borrowing.

(1) HEDGING: Given the stochastic nature of the market size A, retailers would like to adjust their procurement budgets according to the different realizations of A_τ. Specifically, when A_τ is small the initial budget should be enough to procure an optimal amount of product but when A_τ is large, more funds might be needed to procure an optimal amount. A pure hedging strategy is designed to achieve precisely this objective by transferring budget resources from states of nature where they are not needed to states where they are. This means that for a hedging strategy to work effectively one needs some degree of correlation between the retailers' cash-flows and the financial market. In our case, this boils down to requiring that the market demand, i.e. A, is correlated[3] with the financial market.

There are a few equivalent ways in which we can model the financial market and how retailers can construct a hedging strategy. For the purpose of our work, we will take a high-level view and represent retailer i's hedging strategy by a time-τ contingent claim $G_{i\tau}$ that the retailer purchases at time $t = 0$ from a financial institution or market-maker. Alternatively, we can imagine that a financially sophisticated retailer can execute a self-financing trading strategy in the financial market during the time interval $[0, \tau]$ to generate $G_{i\tau}$ by time τ. The mathematical framework that we use to represent the class $\mathcal{G}_\tau$ of contingent claims that the retailers can purchase at time 0 includes: (i) a probability space $(\Omega, \mathbb{Q}, \mathcal{F})$ with expectation operator $\mathbb{E}[\cdot]$ and (ii) a sub σ-algebra $\mathcal{F}_\tau \subseteq \mathcal{F}$ of events representing the publicly available information in the financial market at time τ. We then define

$$\mathcal{G}_\tau := \{G_\tau \in \mathcal{F}_\tau : \mathbb{E}[|G_\tau|] < \infty\}$$

where $\mathbb{E}[G_\tau]$ is the time 0 market price of $G_\tau \in \mathcal{G}_\tau$. In the language of financial economics, $\mathcal{F}_\tau$ is the class of contingent claims that are *attainable* by time τ and $\mathbb{Q}$ is an *equivalent martingale measure* (EMM) or *pricing kernel*. Note that we have implicitly assumed without loss of generality that the risk-free interest rate is identically zero.

If retailer i purchases $G_{i\tau} \in \mathcal{F}_\tau$ at time 0 then his effective budget at time τ is $B + G_{i\tau} - \mathbb{E}[G_{i\tau}]$. Since $G_{i\tau} - \mathbb{E}[G_{i\tau}]$ belongs to $\mathcal{F}_\tau$ and has zero mean, it is therefore convenient to redefine the set of available claims as

$$\mathcal{G}_{0\tau} := \{G_\tau \in \mathcal{F}_\tau : \mathbb{E}[|G_\tau|] < \infty \quad \text{and} \quad \mathbb{E}[G_\tau] = 0\}$$

[3]See Gaur and Seshadri (2005) for some examples in the contexts of apparel, consumer electronics, and home furnishings products.

and to identify a hedging strategy for retailer i by a contingent claim $G_{i\tau} \in \mathcal{G}_{0\tau}$. Under this transformation, retailer i's effective budget at time τ is given by $B + G_{i\tau}$ where $G_{i\tau} \in \mathcal{G}_{0\tau}$.

(2) Borrowing: In addition to hedging their budget constraints, retailers can also borrow in the financial market at time τ in order to increase the amount of inventory they can procure at that time. While we have assumed the risk-free rate is identically zero, we will assume that such borrowing is expensive. Specifically, let r_τ denote the $\mathcal{F}_\tau$-measurable interest-rate at time τ and let $D_{i\tau} \geq 0$ denote the $\mathcal{F}_\tau$-measurable amount borrowed by retailer i at that time. The retailer must then repay $(1 + r_\tau)\, D_{i\tau}$ at time T after the market clears and operating cash flows are realized by the retailers.

It is important to emphasize there is no contradiction in simultaneously assuming a risk-free rate of zero and a (possibly random) costly borrowing rate of $r_\tau > 0$. Specifically, we can view the costly borrowing as reflecting some combination of (i) the (unmodeled) default-risk[4] of the retailers (ii) regulatory requirements whereby banks incur expensive capital charges for lending and (iii) administrative costs and fees associated with debt financing. We note these kinds of frictions are not present to nearly the same extent when it comes to the construction of G_τ. For example (and as mentioned earlier), one can imagine G_τ arising from a dynamic self-financing trading strategy executed by a retailer's corporate treasury function. Such a strategy could easily be conducted via a brokerage account with cheap trading fees and an associated margin account to eliminate default / credit risk. In contrast, raising a lump sum $D_{i\tau}$ is *not* something that can be done via a brokerage account or dynamic trading and always incurs some combination of administrative costs, underwriting fees and credit risk charges.

While borrowing is therefore expensive, it can nonetheless complement the hedging component of a retailer's strategy. To see this, note that the time τ budget constraint will be $w_\tau\, q_{i\tau} \leq B + G_{i\tau} + D_{i\tau}$ for all $\omega \in \Omega$. Borrowing is costly but suppose we have some time τ-measurable event $\mathcal{E}$ in which r_τ is "sufficiently" small. In that case a retailer might choose a claim G_τ where G_τ is large and negative on $\mathcal{E}$. Recalling that we must have $\mathbb{E}[G_\tau] = 0$ this would enable G_τ to be "large" on states in the complement of $\mathcal{E}$. By choosing to borrow in the event $\mathcal{E}$ (where r_τ is small) the retailer can ensure his budget constraint is also satisfied on $\mathcal{E}$ despite the large negative value of G_τ there. The ability to borrow at time τ therefore allows us to satisfy the budget constraint at that time and so it enables a wider set of hedging gains G_τ to be considered. Indeed we will see this type of behavior occur when we solve the Cournot game in Proposition 2 of Section 3.

A few other remarks are in order. First, if we set $r_\tau \equiv 0$ and $D_{i\tau}$ was unconstrained in sign (so lending and borrowing are possible at time τ at a zero interest-rate) then there would be no need for hedging and the problem need only be solved at time τ. In this case the retailers' budgets would also become irrelevant. But of course this is not at all realistic and companies are often (at least in part) budget-constrained due to the fact that raising external finance is often very expensive. Second, we note there is no need to allow for borrowing at time $t = 0$ as the retailers only need to pay the producer at time τ.

[4] In our later numerical examples we will assume a functional form for r_τ that is piecewise linear and (weakly) decreasing in A_τ. We can interpret this via a default-risk narrative whereby the retailers are more likely to default in the event of a poor market, i.e. a market with smaller values of A_τ. These smaller values of A_τ are therefore accompanied by higher values of r_τ.

2.3 The Cournot-Stackelberg Game

We now formulate the retailers' Cournot game assuming the producer has already announced a wholesale price contract w_τ. First note the terminal, i.e. time T, net cash position for retailer i satisfies

$$\begin{aligned}\text{Net-Cash-Position}_i &= (A-(q_{i\tau}+Q_{i\tau-}))\,q_{i\tau}-(1+r_\tau)\,D_{i\tau}+(B+G_{i\tau}+D_{i\tau}-w_\tau q_{i\tau}) \quad (3)\\ &= (A-(q_{i\tau}+Q_{i\tau-})-w_\tau)\,q_{i\tau}-r_\tau\,D_{i\tau}+(B+G_{i\tau})\end{aligned}$$

where the terms on the r.h.s. of (3) correspond to the sales revenue, repayment of debt (principal plus interest), and the budget remaining[5] after paying the producer, respectively. Since $G_{i\tau}$ must have zero expectation and B is a constant, we can therefore[6] formulate the i^{th} retailer's best-response optimization problem as:

$$\begin{aligned}\Pi_{\mathrm{R}_i}(w_\tau) \;&:= \max_{q_{i\tau},G_{i\tau},D_{i\tau}} \mathbb{E}[(A_\tau-(q_{i\tau}+Q_{i\tau-})-w_\tau)\,q_{i\tau}-r_\tau\,D_{i\tau}] \quad (4)\\ \text{subject to}\quad & w_\tau\,q_{i\tau}\le B+G_{i\tau}+D_{i\tau}, \qquad \text{for all } \omega\in\Omega, \quad (5)\\ & q_{i\tau}\ge 0, \qquad D_{i\tau}\ge 0 \quad\text{and}\quad G_{i\tau}\in\mathcal{G}_{0\tau}. \quad (6)\end{aligned}$$

Technically we should have A rather than $A_\tau=\mathbb{E}_\tau[A]$ in (4) but since all the other terms in (4) are $\mathcal{F}_\tau$-measurable, we can use a simple conditioning argument to justify the use of A_τ there.

Let $(q^*_{i\tau}(w_\tau,Q_{i\tau-}), G^*_{i\tau}(w_\tau,Q_{i\tau-}), D^*_{i\tau}(w_\tau,Q_{i\tau-}))$ denote a solution to (4)-(6) for $i=1,\ldots,N$. For such a solution to be an equilibrium, it must also satisfy the fixed-point condition:

$$Q_{i\tau-}=\sum_{j\neq i} q^*_{j\tau}(w_\tau,Q_{j\tau-}) \qquad \text{for all } i=1,\ldots,N \quad\text{and}\quad \text{for all } \omega\in\Omega. \quad (7)$$

Turning to the Stackelberg game, the producer acting as the leader would like to select the wholesale price menu w_τ that maximizes his expected profits taking into account the outcome of the Cournot game in the retailer's market. That is, the producer solves

$$\Pi_{\mathrm{P}} := \max_{w_\tau} \mathbb{E}\big[(w_\tau-c_\tau)\sum_{i=1}^{N} q_{i\tau}(w_\tau)\big] \quad (8)$$

where the $q_{i\tau}$'s are the Cournot equilibrium solution that solve (4) to (7).

Definition 2.1 *A* Cournot-Stackelberg equilibrium *is an $\mathcal{F}_\tau$-measurable wholesale price menu w_τ chosen by the producer and an $\mathcal{F}_\tau$-measurable vector of order quantities and financial strategies $\{(q_{i\tau},G_{i\tau},D_{i\tau}):i=1,\ldots,N\}$ chosen by the retailers such that:*

(i) *$\{(q_{i\tau},G_{i\tau},D_{i\tau}):i=1,\ldots,N\}$ solves the optimization problem (4)-(6) and satisfies the fixed-point condition (7) and*

(ii) *w_τ solves the optimization problem (8).*

[5]The remaining budget is actually "realized" at time τ but we can assume it earns interest between τ and T at the risk-free rate which is zero.

[6]By omitting the constant B from the definition of $\Pi_{\mathrm{R}_i}(w_\tau)$ we can interpret this latter quantity as the retailer's expected profits from operating the business.

In the process of deriving a Cournot-Stackelberg equilibrium for the game, we will make the following assumption.

Assumption 1 (Information)

(a) *The retailers' initial budget B and the probability distribution of A are common knowledge to all players. More generally, we are assuming a common knowledge framework in which all parameters of the model are known to all agents.*

(b) *The only information regarding the market demand A that is revealed to the retailers during the time interval $[0, \tau]$ is the information contained in $\mathcal{F}_\tau$, i.e. the information in the financial markets up to time τ.*

Assumption 1 (a) is required to ensure that our Cournot-Stackelberg game is tractable. Assumption 1 (b) is convenient for the following reason. Suppose each retailer had access to additional information (private or public) related to the market size A beyond the information revealed by the financial market by time τ. If $G_{i\tau}$ is obtained via a self-financing trading strategy then this additional information could be used as part of the trading strategy in which case we could no longer assume that each $G_{i\tau}$ is $\mathcal{F}_\tau$-measurable. But solving for an optimal $G_{i\tau}$ in this case would require the solution of a difficult optimal control problem and ultimately would lead to the Cournot-Stackelberg game being hopelessly intractable. If on the other hand, we assumed that $G_{i\tau}$ is simply purchased at time $t = 0$ from a financial institution then it would still make sense to insist that it be $\mathcal{F}_\tau$-measurable regardless of whether or not additional non-financial information was available. On the other hand, a real difficulty would arise with the debt component $D_{i\tau}$. In particular, there would be no reason to insist that $D_{i\tau}$ be $\mathcal{F}_\tau$-measurable if additional non-financial information on A was available at time τ. But this would add unnecessary complications to the model which is already sufficiently challenging to solve. For this reason we will proceed with Assumption 1 in the sequel.

2.4 Further Model Discussion

Before proceeding to analyze the Cournot and Cournot-Stackelberg games a number of further clarifying remarks are in order.

1. In this model the producer does not trade in the financial markets because, being risk-neutral and not restricted by a budget constraint, he has no incentive to do so.

2. A potentially valid criticism of this model is that, in practice, a retailer is often a small entity and may not have the ability to trade in the financial markets. There are a number of responses to this. First, we use the word 'retailer' in a loose sense so that it might in fact represent a large entity. For example, an airline purchasing aircraft is a 'retailer' that certainly does have access to the financial markets. Second, it is becoming ever cheaper and easier for even the smallest 'player' to trade in the financial markets and many of them do so routinely to hedge interest rate risk, foreign exchange rate risk etc.

3. Another potentially valid criticism of this framework is that the class of contracts is too complex. In particular, by only insisting that w_τ is $\mathcal{F}_\tau$-measurable we are permitting wholesale price contracts that might be too complicated to implement in practice. If this is the case then we can easily simplify the set of feasible contracts. By using appropriate conditioning arguments, for example, it would be straightforward to impose the tighter restriction that w_τ be $\sigma(X_\tau)$-measurable instead where $\sigma(X_\tau)$ is the σ-algebra generated by X_τ, the time τ value of some observable financial index or security. More importantly, the insights we gain from our model should also apply to other settings where we impose more restrictions on the class of contracts that are considered. Indeed in our numerical example of Section 4.1 we will compare the Cournot-Stackelberg equilibrium for a general price contract (where w_τ just needs to be $\mathcal{F}_\tau$-measurable) with the corresponding Cournot-Stackelberg equilibrium for a price contract where w_τ is restricted to be a constant price. We will see there that the two equilibria are qualitatively very similar.

3 The Cournot Equilibrium

We now derive the Cournot equilibrium for the model we described in Section 2 but taking the price menu w_τ as fixed. While the Cournot equilibrium is required to solve for the full Cournot-Stackelberg equilibrium, it is interesting in its own right. For example, in some circumstances there may not be a Stackelberg leader with price control (e.g., if there are many producers from whom the retailers can procure), and taking w_τ as exogenous and fixed may indeed be more appropriate. Moreover, the solution of the Cournot equilibrium will help us provide some intuition for the full Cournot-Stackelberg equilibrium of Section 4.

In order to gain some insight into the problem we will first consider the case where the retailers' hedging strategies $G_{i\tau}$ are fixed and then solve for the optimal ordering levels $q^*_{i\tau}(w_\tau, G_{i\tau})$ and borrowing $D^*_{i\tau}(w_\tau, G_{i\tau})$ as a function of w_τ and $G_{i\tau}$. This is the subject of Proposition 1 and Corollary 1. Afterwards, we will solve for the full Cournot equilibrium (where the retailers also optimize over their hedging strategies) in Proposition 2.

Suppose then each retailer i selects a feasible hedging strategy $G_{i\tau}$ for $i \in [N] := \{1, \ldots, N\}$. Then, the available budget of retailer i at time τ is $B_{i\tau} = B + G_{i\tau}$. Recall that $Q_{i\tau-} := \sum_{j \neq i} q_{j\tau}$ is the cumulative orders of all the retailers excluding retailer i. Given the budget $B_{i\tau}$ and the wholesale price menu w_τ, retailer i's best response ordering and borrowing strategies at time τ are determined by solving:

$$\begin{aligned}
\Pi_{\mathrm{R}_i|\tau}(w_\tau, B_{i\tau}, Q_{i\tau-}) \quad &= \quad \max_{q_{i\tau}, D_{i\tau}} \left(A_\tau - (q_{i\tau} + Q_{i\tau-}) - w_\tau\right) q_{i\tau} - r_\tau\, D_{i\tau} && (9)\\
\text{subject to} \qquad & w_\tau\, q_{i\tau} \leq B_{i\tau} + D_{i\tau}, \qquad \text{for all } \omega \in \Omega, && (10)\\
& q_{i\tau} \geq 0 \quad \text{and} \quad D_{i\tau} \geq 0. && (11)
\end{aligned}$$

We can easily solve this optimization problem via the first-order KKT conditions and the following lemma provides the solution.

Lemma 1 *An optimal solution to (9)-(11) is given by*

$$q^*_{i\tau} = \frac{(A_\tau - Q_{i\tau-} - (1 + \lambda_{i\tau})\, w_\tau)^+}{2} \qquad \textit{and} \qquad D^*_{i\tau} = (w_\tau\, q^*_{i\tau} - B_{i\tau})^+, \quad i \in [N] \qquad (12)$$

where $\lambda_{i\mathcal{T}} = \min\left\{r_{\mathcal{T}}, \alpha_{i\mathcal{T}}^{+}\right\}$ *and* $\alpha_{i\mathcal{T}}$ *is the solution to the equation*

$$B_{i\mathcal{T}} = \frac{w_\tau\left(A_\tau - Q_{i\mathcal{T}-} - (1+\alpha_{i\mathcal{T}})\, w_\tau\right)^{+}}{2}$$

with $x^{+} := \max(0, x)$.

In the statement of Lemma 1, $\lambda_{i\mathcal{T}}$ is the Lagrange multiplier for the budget constraint in (10) and measures retailer's i marginal value of capital given its budget $B_{i\mathcal{T}}$. The retailer's ability to borrow at a rate $r_{\mathcal{T}}$ implies that at optimality we must have $\lambda_{i\mathcal{T}} \leq r_{\mathcal{T}}$. Similarly, it is not hard to see that $D^{*}_{i\mathcal{T}} > 0$ only if $\lambda_{i\mathcal{T}} = r_{\mathcal{T}}$.

Using the best-response strategy in Lemma 1 we can now characterize the Cournot equilibrium in the retailers' market for a given wholesale price menu w_τ and vector of budgets $\{B_{i\mathcal{T}}\}_{i\in[N]}$.

Proposition 1 (Cournot Equilibrium with Fixed Hedging Strategies)
Given a wholesale price menu w_τ *and a vector of budgets* $\{B_{i\mathcal{T}}\}_{i\in[N]}$ *for the retailers, the equilibrium cumulative orders* $Q^{*}_{\mathcal{T}}$ *of all retailers is the unique solution of the fixed-point equation:*

$$Q_{\mathcal{T}} = \sum_{i=1}^{N} \max\left\{(A_\tau - Q_{\mathcal{T}} - (1+r_{\mathcal{T}})\, w_\tau)^{+},\ \min\left\{(A_\tau - Q_{\mathcal{T}} - w_\tau)^{+},\ \frac{B_{i\mathcal{T}}}{w_\tau}\right\}\right\}.$$

The equilibrium order quantity and level of debt of retailer $i \in [N]$ *are given by*

$$q^{*}_{i\mathcal{T}} = \max\left\{(A_\tau - Q^{*}_{\mathcal{T}} - (1+r_{\mathcal{T}})\, w_\tau)^{+},\ \min\left\{(A_\tau - Q^{*}_{\mathcal{T}} - w_\tau)^{+},\ \frac{B_{i\mathcal{T}}}{w_\tau}\right\}\right\}$$

and $D^{*}_{i\mathcal{T}} = (w_\tau\, q^{*}_{i\mathcal{T}} - B_{i\mathcal{T}})^{+}$, *respectively.*

The following corollary considers the special case in which the retailers use a symmetric hedging strategy so that $B_{i\mathcal{T}} = B_\tau$ which in general is random. (In Proposition 2 below we will show that in equilibrium the retailers do indeed select a symmetric equilibrium.)

Corollary 1 *Suppose* $B_{i\mathcal{T}} = B_\tau$ *for all* $i \in [N]$, *then the equilibrium in Proposition 1 reduces to*

$$q^{*}_{\mathcal{T}} = \frac{1}{w_\tau} \max\left\{\underline{B}(w_\tau, r_{\mathcal{T}}), \min\left\{\overline{B}(w_\tau), B_\tau\right\}\right\} \qquad \textit{and} \qquad D^{*}_{\mathcal{T}} = \left(\underline{B}(w_\tau, r_{\mathcal{T}}) - B_\tau\right)^{+},$$

where

$$\underline{B}(w_\tau, r_{\mathcal{T}}) := \frac{w_\tau\left(A_\tau - (1+r_{\mathcal{T}})\, w_\tau\right)^{+}}{N+1} \qquad \textit{and} \qquad \overline{B}(w_\tau) := \frac{w_\tau\left(A_\tau - w_\tau\right)^{+}}{N+1}.$$

Figure 1 illustrates the result in the corollary. From the left panel we see that the retailers only raise debt to finance their operations when their budget B_τ is below the threshold $\underline{B}(w_\tau, r_{\mathcal{T}})$. Furthermore, in this case they raise an amount $D^{*}_{\mathcal{T}}$ so as to bring their available budget $B_\tau + D^{*}_{\mathcal{T}}$ exactly to the level $\underline{B}(w_\tau, r_{\mathcal{T}})$. In the intermediate region $\underline{B}(w_\tau, r_{\mathcal{T}}) \leq B_\tau \leq \overline{B}(w_\tau)$, the retailers choose not to borrow despite the fact that their budget constraint is binding since $w_\tau\, q^{*}_{\mathcal{T}} = B_\tau$. Intuitively, in this region the marginal return to borrowing is lower than the cost of debt $r_{\mathcal{T}}$. Finally, when the retailers budget is sufficiently high, namely when $B_\tau \geq \overline{B}(w_\tau)$, the budget constraint is no longer binding (due possibly to hedging) and the retailers' ordering quantity $q^{*}_{\mathcal{T}}$ reaches the

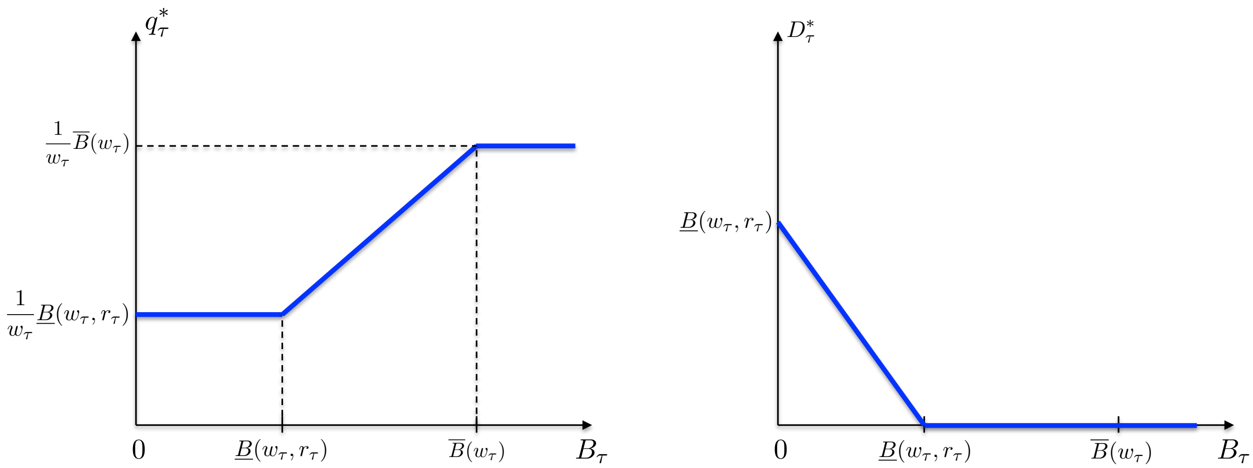

Figure 1: Ordering (q^*_τ) and borrowing (D^*_τ) strategies for the Cournot equilibrium as a function of B_τ.

unconstrained level $\overline{B}(w_\tau, r_\tau)/w_\tau$. It is also worth noting that access to borrowing bounds the value of q^*_τ from below at the level $\underline{B}(w_\tau, r_\tau)/w_\tau$. It follows that q^*_τ is only sensitive to the retailers' budget B_τ in the intermediate region $\underline{B}(w_\tau, r_\tau) \leq B_\tau \leq \overline{B}(w_\tau)$.

We now turn to the question of characterizing the complete Cournot equilibrium in the retailers' market where they also optimize over their hedging strategies $G^*_{i\mathcal{T}}$. We will see that the insight gained from Corollary 1 and Figure 1 continue to apply with just one difference. The results of Proposition 1 and Corollary 1 hold on a state-by-state basis so that $\underline{B}(w_\tau, r_\mathcal{T})$ and $\overline{B}(w_\tau)$ are random variables. However, when we also optimize for the hedging strategies $G^*_{i\mathcal{T}}$ in the equilibrium, $\underline{B}(w_\tau, r_\mathcal{T})$ and $\overline{B}(w_\tau)$ are replaced by expectations and hence become scalar quantities.

In the statement of Proposition 2 below we use the following definition.

Definition 3.1 *Let $\lambda^*(B) = \max\{0, \min\{\underline{r}_\mathcal{T}, \alpha^*(B)\}\}$, where $\underline{r}_\mathcal{T} := \inf_{\omega\in\Omega}\{r_\mathcal{T}\}$ is the minimum possible interest rate at which the retailers can borrow and let $\alpha^*(B)$ be the unique solution of the equation*

$$B = \frac{1}{N+1}\,\mathbb{E}\left[w_\tau\left(A_\tau - w_\tau\,(1+\alpha)\right)^+\right]. \tag{13}$$

In our next proposition, $\lambda^*(B)$ plays the role of the Lagrange multiplier of the retailer's budget constraint and as such measures the marginal value of capital in equilibrium. Note that $\lambda^*(B)$ is deterministic –as opposed to the stochastic Lagrange multipliers $\lambda_{i\mathcal{T}}$ derived in Lemma 1 where the hedging strategies were given exogenously.

Proposition 2 (Full Cournot Equilibrium)
Let $w_\mathcal{T}$ be a fixed wholesale price menu offered by the producer. There is a unique equilibrium in terms of retailer ordering quantities and this equilibrium is symmetric. Specifically retailer i orders an amount equal to

$$q^*_{i\mathcal{T}} = q^*_\mathcal{T} := \frac{1}{N+1}\left(A_\tau - w_\tau\,(1+\lambda^*(B))\right)^+ \tag{14}$$

and the total output in the market is given by $Q^*_\mathcal{T} = N\,q^*_\mathcal{T}$. In addition, retailer i's financial strategy, i.e., hedging and borrowing, can be summarized via three cases depending on the value of B:

- Case 1: *If* $B \geq \mathbb{E}\big[\overline{B}(w_\tau)\big]$ *then* $\alpha^*(B) \leq 0$ *and* $\lambda^*(B) = 0$.

  In this case, the retailer uses no debt, i.e. $D^*_{i\mathcal{T}} = 0$*, and there are infinitely many hedging strategies* $G_{i\mathcal{T}}$ *that can implement the optimal ordering quantity* $q^*_{i\mathcal{T}}$*. One particular choice is*

$$G^*_{i\mathcal{T}} = (w_\tau\, q^*_\tau - B) \cdot \begin{cases} \delta_\tau & \text{if } \omega \in \mathcal{X}_\tau \\ 1 & \text{if } \omega \in \mathcal{X}^c_\tau \end{cases}$$

  where $$\delta_\tau := \frac{\int_{\mathcal{X}^c_\tau} [w_\tau\, q^*_\tau - B]\, \mathrm{d}\mathbb{Q}}{\int_{\mathcal{X}_\tau} [B - w_\tau\, q^*_\tau]\, \mathrm{d}\mathbb{Q}}, \qquad \mathcal{X}_\tau := \{\omega \in \Omega \,:\, B \geq w_\tau\, q^*_\tau\} \quad \text{and} \quad \mathcal{X}^c_\tau := \Omega - \mathcal{X}_\tau.$$

- Case 2: *If* $\mathbb{E}\big[\underline{B}(w_\tau, \underline{r}_\tau)\big] \leq B \leq \mathbb{E}\big[\overline{B}(w_\tau)\big]$ *then* $0 \leq \alpha^*(B) \leq \underline{r}_\tau$ *and* $\lambda^*(B) = \alpha^*(B)$.

  In this case, retailer i *does not raise any debt, i.e.* $D^*_{i\mathcal{T}} = 0$*, and uses the hedging strategy* $G^*_{i\mathcal{T}} = w_\tau\, q^*_{i\mathcal{T}} - B$.

- Case 3: *If* $B \leq \mathbb{E}\big[\underline{B}(w_\tau, \underline{r}_\tau)\big])$ *then* $\alpha^*(B) \geq \underline{r}_\tau$ *and* $\lambda^*(B) = \underline{r}_\tau$.

  Define the event $\mathcal{E}_\tau := \{r_\tau = \underline{r}_\tau\}$*. Then the retailer's borrowing strategy is such that it only raises debt on* $\mathcal{E}$ *and there are infinitely many possible borrowing strategies* $D^*_{i\mathcal{T}}$ *that the retailer can use with the only requirement being that* $\mathbb{E}[D^*_{i\mathcal{T}}] = \mathbb{E}[w_\tau\, q^*_{i\mathcal{T}}] - B$*. One specific choice that borrows uniformly on* $\mathcal{E}_\tau$ *is*

$$D^*_{i\mathcal{T}} = \left(\frac{\mathbb{E}[w_\tau\, q^*_{i\mathcal{T}}] - B}{\mathbb{P}(\mathcal{E}_\tau)}\right) \mathbb{1}(\mathcal{E}_\tau). \tag{15}$$

  The retailer's optimal hedging strategy satisfies $G^*_{i\mathcal{T}} = w_\tau\, q^*_{i\mathcal{T}} - B - D^*_{i\mathcal{T}}$.

Finally, in equilibrium, the expected payoffs of the producer and each retailer are given by

$$\Pi_{\mathrm{P}} = \mathbb{E}\Big[(w_\tau - c_\tau)\, Q^*_\tau\Big] \qquad \text{and} \qquad \Pi_{\mathrm{R}} = \mathbb{E}\left[(A_\tau - Q^*_\tau - w_\tau)\, q^*_\tau\right] - \underline{r}_\tau \Big(\mathbb{E}[\underline{B}(w_\tau, \underline{r}_\tau)] - B\Big)^+.$$

Several remarks regarding Proposition 2 are in order. First, note that the conditions determining which of the three cases apply simply reflect the value of $\alpha^*(B)$ as determined by (13). Case 1 corresponds to the case in which the retailer's initial budget B is large enough so that it can order an amount $q^*_{i\mathcal{T}}$ that is equal to the optimal unconstrained level, i.e. the level that would be ordered if the initial budgets were infinite. In this case $\lambda^*(B) = 0$ and so $\mathbb{E}\big[w_\tau\,(A_\tau - w_\tau)^+\big]/(N+1)$ represents the threshold for the value of B above which this unconstrained solution can be achieved. Note that this case includes the situation where B is sufficient to procure the optimal quantity $q^*_{i\mathcal{T}}$ on every path in which case $\mathcal{X}^c_\tau = \emptyset$ and so $\delta_\tau = 0$. It is worth emphasizing, however, that in general under Case 1 it is not the case that B is sufficient to procure the optimal quantity $q^*_{i\mathcal{T}}$ on every path. Rather, it is the retailer's ability to hedge and reallocate its budget across states that leads to this unconstrained solution. Note also that the retailer does not use any debt in this case.

In Case 2, we have $0 \leq \lambda^*(B) \leq \underline{r}_\tau$. Now the initial budget is no longer sufficient to sustain the optimal unconstrained ordering quantity of Case 1. It follows that the retailers order less (than in Case 1) and use all of their effective budgets on a pathwise basis. As with Case 1, however, the retailers do hedge but do not use any debt financing.

Finally, Case 3 corresponds to the case in which $\lambda^*(B) = \underline{r}_\tau$ and the retailers' budgets are sufficiently small that (costly) borrowing is worthwhile. In this case $\mathbb{E}\big[w_\tau\,(A_\tau - w_\tau\,(1 + \underline{r}_\tau))^+\big]/(N+1)$ is the budget threshold below which raising debt becomes optimal and this is done only on those states where the cost of debt is cheapest. An immediate consequence of the retailers' ability to use financial hedging to minimize the cost of debt across states of the world is that in the Cournot equilibrium their borrowing decisions depend exclusively on their initial budget B rather than on their budget $B + G^*_{i\tau}$ available at time τ when borrowing decisions are actually made.

Remark 2 *The proof of Proposition 2 shows that $\lambda^*(B)$ is the optimal Lagrange multiplier for the hedging constraint $\mathbb{E}[G_{i\tau}] = 0$ which at optimality is also equal to the Lagrange multipliers of the budget constraints $w_\tau \, q_{i\tau} \leq B + G_{i\tau} + D_{i\tau}$. This means the budget constraints across different states have the same optimal Lagrange multiplier so that in equilibrium the hedging gains $G_{i\tau}$ are chosen so as to equalize the marginal costs of the effective budget constraints across the different states. It is also clear from the statement of Proposition 2 that $0 \leq \lambda^*(B) \leq \underline{r}_\tau$. That $\lambda^*(B)$ cannot exceed $\underline{r}_\tau$ is intuitively clear as the retailers always have the option of effectively[7] borrowing at a rate of $\underline{r}_\tau$ in any state $\omega \in \Omega$. Finally we note that $\lambda^*(B)$ is a continuous function of B which means that the optimal ordering quantities $q^*_{i\tau}$ are also continuous functions of B for a fixed price menu w_τ. This will be relevant when we solve for the full Cournot-Stackelberg equilibrium in Section 4 where we will see that the optimal ordering quantities are potentially discontinuous in B.*

3.1 An Illustrative Example

In order to demonstrate some of the key features of the Cournot equilibrium and the value of production postponement, we consider a numerical example with the following characteristics: (i) there are $N = 5$ retailers (ii) the market potential A_τ is uniformly distributed with $A_\tau \sim U[\underline{A}_\tau, \bar{A}_\tau]$ with $\underline{A}_\tau = 100$ and $\bar{A}_\tau = 300$ (iii) the per-unit production cost is constant with $c_\tau = 80$ (iv) the cost of debt r_τ is a piece-wise linear non-increasing function of the market potential A_τ given by $r_\tau = \max\{0.7 - 0.002\, A_\tau \, ; \, 0.2\}$ so that $\underline{r}_\tau = 0.2$ (v) the producer uses a constant wholesale price policy with $w_\tau = 1.9\, c_\tau$, i.e. the producer sets a constant margin of 90%.

Figure 1 illustrates the results of Proposition 2. The left panel in Figure 1 depicts the producer's expected payoff as a function of the retailers' initial budget B while the right panel depicts the expected payoff of a single retailer as a function of B. To assess the value produced by the financial markets, each panel includes four[8] curves: (1) the red curves (with dots) correspond to the case in which retailers have no access to the financial markets (neither hedging nor borrowing) (2) the black curves (with diamonds) correspond to the case where the retailers can hedge but don't have access to the costly debt market (3) the blue curves (with squares) correspond to the case in which the retailers can borrow but can't hedge and (4) the green curves correspond to the case in which the retailers have full access to the financial markets and therefore can hedge and raise costly debt.

[7] While they would only borrow on the event $\mathcal{E} = \{r_\tau = \underline{r}_\tau\}$, the ability to hedge means they could borrow on $\mathcal{E}$ and then shift these borrowings via $G^*_{i\tau}$ to any other state in which they wish to borrow. On a related note, it's clear from Case 3 of Proposition 2 that the retailers will want to borrow on the event $\mathcal{E} := \{r_\tau = \underline{r}_\tau\}$ if their budgets are sufficiently small. If $\mathbb{P}(\mathcal{E}_\tau) = 0$, however, (as would be the case, for example, in most continuous-time models of r_τ) then this would require borrowing an infinite amount on the event $\mathcal{E}$; see (15). This of course would be infeasible in practice but we could resolve this issue by approximating $\mathcal{E}$ with an event $\widehat{\mathcal{E}}_\tau := \{r_\tau \leq \underline{r}_\tau + \epsilon\}$ for some fixed but small $\epsilon > 0$ and then borrowing on $\widehat{\mathcal{E}}_\tau$ rather than on $\mathcal{E}_\tau$.

[8] Appendix B derives the Cournot equilibria for the three benchmark cases where the retailers don't have full access to the financial markets. Proposition 2 provides the Cournot equilibrium for the case where they do have full access.

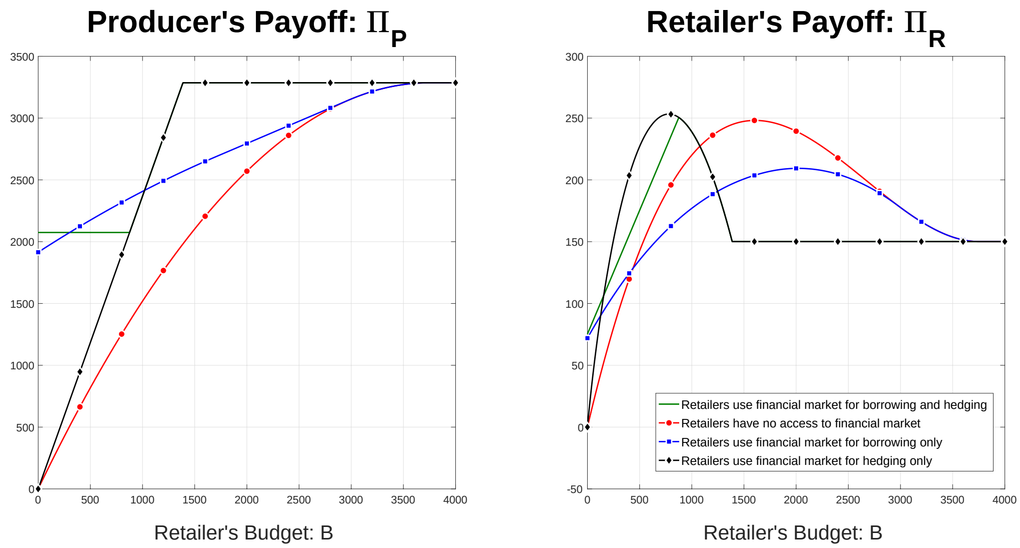

Figure 2: Producer's payoff $\Pi_{\mathsf{P}|\tau}$ (left panel) and retailer's payoff $\Pi_{\mathsf{R}|\tau}$ (right panel) as a function of B.
DATA: $N = 5$, $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $r_\tau = \max\{0.7 - 0.002\, A_\tau\,;\, 0.2\}$ and $w_\tau = 1.9\, c_\tau$.

A number of remarks are in order. First, while the producer's expected payoff is uniformly non-decreasing in the retailers' budget B (in all four cases), the retailers' payoffs are decreasing for some values of B. This apparent anomaly is driven by the non-cooperative nature of the Cournot equilibrium in the retailers' market. Indeed, as B increases each retailer increases its orders, thereby placing more inventory on the market and pushing down the retail price $P = A - Q$.

Another difference between the expected payoffs of the producer and the retailers is that independently of the retailers' budget B the producer is always better off if the retailers have some access to the financial market (to borrow, hedge or both) while it is possible for some values of B that the retailers are better off is they have no access at all to the financial market. As we mentioned before, this can be explained by noting that access to the financial markets enable the retailers to make better use of their budgets allowing them to increase their orders when market conditions are more favorable and thereby pushing the retail prices down.

It is also interesting that for most values of B the producer is better off if the retailers use the financial markets to borrow and hedge. However, there is a range of B values in which the producer is better off if the retailers can only use the financial markets to borrow. The reason is that in this case hedging allows retailers to better use their budget and reduce the average order size placed to the producer. On the retailers' side, their preferences over the four modes of financing in Figure 2 vary significantly as a function of B. The only consistent pattern is that the retailers' expected payoff with only borrowing are dominated by at least one of the other three modes of financing.

3.2 The Value of Production Postponement in the Cournot Equilibrium

We now turn to the question of measuring the value of production postponement by measuring the difference between the payoffs that the producer and retailers obtain from the Cournot equilibrium in Proposition 2 and the payoffs they'd obtain if the procurement orders had to be placed at time $t = 0$. We can recover the Cournot equilibrium for this case by setting $\tau = 0$ so that the random variable A_τ is replaced by the constant $A_0 = \mathbb{E}[A]$. In this case the retailers have no time to trade in the financial markets and so dynamic hedging is not possible. (It's easy to confirm that Proposition 2 yields $G^*_{i\tau} \equiv 0$ in this case.) On the other hand, retailers can still borrow and they do so at the interest rate r_0 available at time 0. In practice, we would expect $r_0 \geq \underline{r}_\tau := \inf_{\omega\in\Omega}\{r_\tau\}$ for any fixed $\tau > 0$ because r_τ is stochastic and so it's reasonable[9] to assume that at least in some states we will have $r_\tau < r_0$. In contrast, we expect c_0, the production cost at time 0, to be lower than c_τ for $\tau > 0$ as the producer has more time to optimize production and logistics at time 0. To capture these conditions, we let $c_\tau = \alpha_\tau\, c_0$ and $(1 + \underline{r}_\tau) = \beta_\tau\,(1 + r_0)$, where α_τ and β_τ are monotonic non-decreasing and non-increasing functions of τ, respectively, and where $\beta_\tau \leq 1 \leq \alpha_\tau$.

Figure 3 depicts the relative value of production postponement for the producer (panels in left column) and the retailers (panels in right column) in the α_τ vs. β_τ space for three values of the retailers' initial budget: $B = 50$ (top row), $B = 500$ (middle row) and $B = 2000$ (bottom row). The relative value of production postponement is defined as the ratio of the expected payoffs of an agent with postponement to the expected payoffs without postponement. For example, if Π_{P} is the producer's expected payoff with production postponement derived in Proposition 2 and $\Pi_{\mathrm{P}|0}$ is the producer's expected payoffs without postponement, then the relative value of production postponement for the producer is equal to $\Pi_{\mathrm{P}}/\Pi_{\mathrm{P}|0}$.

Figure 3 reveals that the value of postponement can be quite significant for small value of B (top row). In this example, postponement can increase the producer's and retailers' payoffs by as much as a factor of 15 and 5, respectively, when $B = 50$, α_τ is close to 1 and β_τ is close to 0.8. It's also worth noting that in this case (where B is relatively low) the value of postponement is sensitive to both α_τ and β_τ. On the other hand, for larger values of B (bottom row), the value of postponement for both agents seems to be more sensitive to β_τ and less sensitive to α_τ. In addition, when B is large, postponement can have a negative effect on the producers' payoff if β_τ is close to one.

4 The Cournot-Stackelberg Equilibrium

We now turn to solving the full Cournot-Stackelberg equilibrium. The retailers' Cournot equilibrium in Proposition 2 defines the demand function Q^*_τ that the producer will face as a function of the wholesale price menu w_τ that he selects. Hence, the optimization problem that the producer must solve may be formulated as

$$\Pi_{\mathrm{P}} = \max_{w_\tau} \;\; \frac{N}{N+1}\; \mathbb{E}\Big[(w_\tau - c_\tau)\;(A_\tau - w_\tau\,(1+\lambda^*))^+\Big] \tag{16}$$

$$\text{subject to } \lambda^* = \max\Big\{0, \min\{\underline{r}_\tau, \alpha^*\}\Big\}, \text{ where } \alpha^* \text{ solves } \mathbb{E}\left[w_\tau\, \frac{(A_\tau - w_\tau\,(1+\alpha^*))^+}{(N+1)}\right] = B. \tag{17}$$

To solve this optimization problem, it's convenient to introduce the thresholds

$$\begin{aligned}
\overline{\overline{B}}_\tau &:= \frac{1}{4(N+1)} \sup_{\omega\in\Omega}\Big\{(A_\tau^2 - c_\tau^2)^+\Big\} \\
\overline{B}_\tau &:= \frac{1}{4(N+1)}\, \mathbb{E}\Big[(A_\tau^2 - c_\tau^2)^+\Big]
\end{aligned}$$

[9]It's also true that at time 0 there is more uncertainty / risk regarding the retailers' market potential A_T. To the extent that the cost of borrowing reflects this increased risk, this provides an additional reason to expect $r_0 \geq \underline{r}_\tau$.

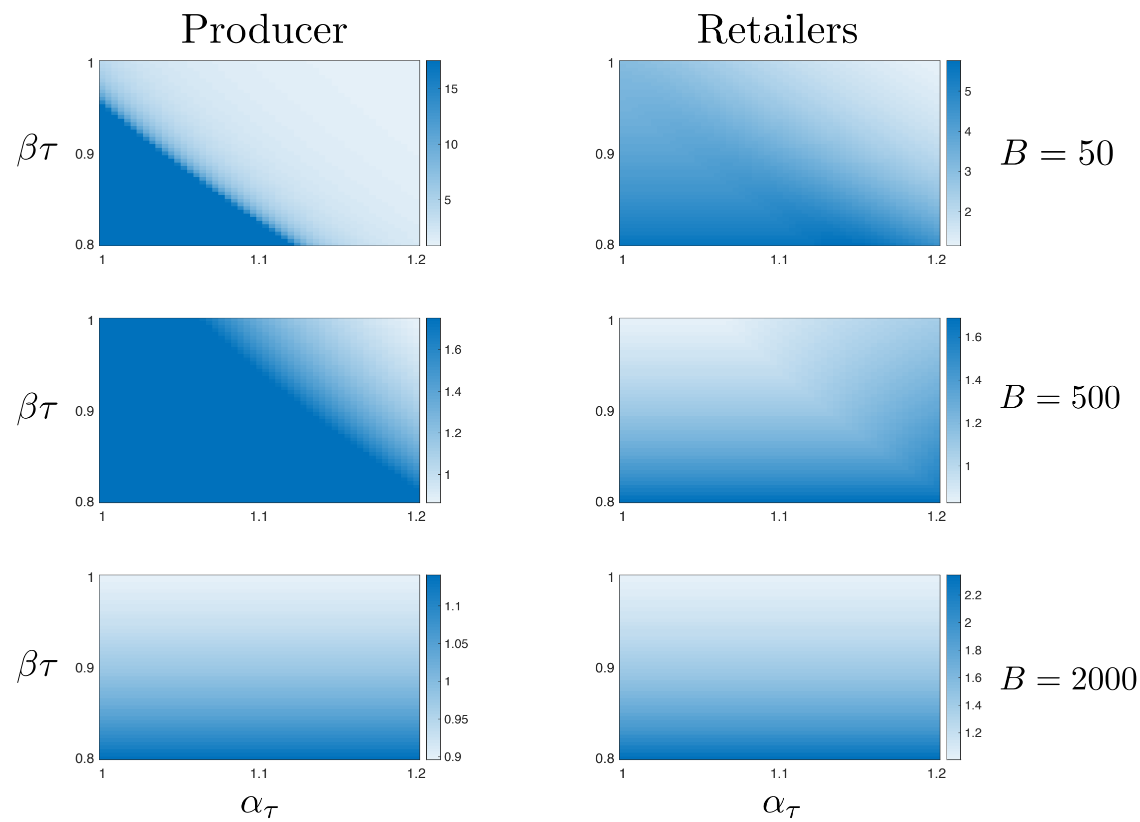

Figure 3: Value of production postponement for the producer $\Pi_{\mathrm{P}}/\Pi_{\mathrm{P}|0}$ (left panels) and the retailers $\Pi_{\mathrm{R}}/\Pi_{\mathrm{R}|0}$ (right panels) in the α_τ vs. β_τ space for three values of the retailers initial budget: $B = 50$ (top row), $B = 500$ (middle row) and $B = 2000$ (bottom row).
DATA: $N = 5$, $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $r_\tau = \max\{0.7 - 0.002\, A_\tau\, ; 0.2\}$, $w_\tau = 1.9\, c_\tau$, $c_0 = c_\tau/\alpha_\tau$ and $r_0 = (1 + \underline{r}_\tau)/\beta_\tau - 1$

and to let $\underline{B}_\tau$ denote the solution of the equation

$$\frac{1}{1+\underline{r}_\tau}\,\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] = 4\,(N+1)\,\underline{B} - 2\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(\underline{B})\,c_\tau\right)^+\right] \tag{18}$$

in B where

$$\phi^*(B) := \min\Big\{\phi \geq 1 \,:\, \mathbb{E}\Big[(A_\tau^2 - \phi^2\, c_\tau^2)^+\Big] \leq 4\,(N+1)\,B\Big\}. \tag{19}$$

We prove the existence and uniqueness of $\underline{B}_\tau$ in Appendix A. The following proposition fully characterizes the Cournot-Stackelberg equilibrium.

Proposition 3 (Cournot-Stackelberg Equilibrium)
We identify four regions depending on the value of the retailers' initial budget B.

- Region I: Unconstrained solution with no need for hedging or borrowing

  Suppose $B \geq \overline{\overline{B}}_\tau$. Then in equilibrium the producer's wholesale price and the retailers' individual ordering quantities satisfy

$$w_\tau^* = \frac{A_\tau + c_\tau}{2} \qquad \textit{and} \qquad q_\tau^* = \frac{1}{N+1}\left(\frac{A_\tau - c_\tau}{2}\right)^+.$$

In addition, the retailers do not use the financial markets and $G^*_\tau = D^*_\tau = 0$. *The expected payoffs of the producer and retailers satisfy*

$$\Pi_{\mathrm{P}} = \frac{N}{4\,(N+1)}\,\mathbb{E}\left[\left((A_\tau - c_\tau)^+\right)^2\right] \qquad \text{and} \qquad \Pi_{\mathrm{R}} = \frac{1}{4\,(N+1)^2}\,\mathbb{E}\left[\left((A_\tau - c_\tau)^+\right)^2\right]. \quad (20)$$

- Region II: Unconstrained solution with hedging but no need for borrowing

  Suppose $\overline{B}_\tau \le B < \overline{\overline{B}}_\tau$. *Then in equilibrium the producer's wholesale price and the retailers' individual ordering quantities satisfy*

$$w^*_\tau = \frac{A_\tau + c_\tau}{2} \qquad \text{and} \qquad q^*_\tau = \frac{1}{N+1}\left(\frac{A_\tau - c_\tau}{2}\right)^+.$$

  In this case the retailers do not borrow, i.e. $D^*_\tau = 0$, *and use the financial market only to hedge. Furthermore, there are infinitely many feasible hedging strategies* G_τ *that a retailer can use to implement the optimal ordering quantity* q^*_τ. *One particular choice is given by*

$$G^*_\tau = \begin{cases} w^*_\tau q^*_\tau - B, & B < w^*_\tau q^*_\tau \\ (w^*_\tau q^*_\tau - B)\,\frac{\mathbb{E}[(w^*_\tau q^*_\tau - B)^+]}{\mathbb{E}[(B - w^*_\tau q^*_\tau)^+]}, & B \ge w^*_\tau q^*_\tau. \end{cases}$$

  The expected payoffs of the producer and the retailers are given by the expressions in (20).

- Region III: Constrained solution with hedging but no need for borrowing

  Suppose $\underline{B}_\tau \le B < \overline{B}_\tau$. *Then in equilibrium the producer's wholesale price and the retailers' individual ordering quantities satisfy*

$$w^*_\tau = \frac{A_\tau + \phi^*(B)\,c_\tau}{2} \qquad \text{and} \qquad q^*_\tau = \frac{1}{N+1}\left(\frac{A_\tau - \phi^*(B)\,c_\tau}{2}\right)^+$$

  where $\phi^*(B)$ *is defined in (19). In this case the retailers do not need to borrow, i.e.* $D^*_\tau = 0$, *and use the financial markets only to hedge with* $G^*_\tau = w^*_\tau q^*_\tau - B$.

  The expected payoffs of the producer and the retailers are given by

$$\Pi_{\mathrm{P}} = N\,B - \frac{N}{2\,(N+1)}\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(B)\,c_\tau\right)^+\right] \quad \text{and} \quad \Pi_{\mathrm{R}} = \frac{\mathbb{E}\left[\left((A_\tau - \phi^*(B)\,c_\tau)^+\right)^2\right]}{4\,(N+1)^2}.$$

- Region IV: Constrained solution with hedging and borrowing

  Suppose $B < \underline{B}_\tau$. *Then in equilibrium the producer's wholesale price and the retailers' individual ordering quantities satisfy*

$$w^*_\tau = \frac{A_\tau + (1+\underline{r}_\tau)\,c_\tau}{2\,(1+\underline{r}_\tau)} \qquad \text{and} \qquad q^*_\tau = \frac{1}{N+1}\left(\frac{A_\tau - (1+\underline{r}_\tau)\,c_\tau}{2}\right)^+.$$

  In this case the retailers use the financial market to both hedge and borrow. Their borrowing strategy is such that they only borrow on $\mathcal{E} = \{r_\tau = \underline{r}_\tau\}$. *There are infinitely many possible borrowing strategies* D^*_τ *that a retailer can use with the only requirement being that* $\mathbb{E}[D^*_\tau] = \mathbb{E}[w^*_\tau\, q^*_\tau] - B$. *One specific choice that borrows uniformly on* $\mathcal{E}$ *is*

$$D^*_\tau = \left(\mathbb{E}\left[w^*_\tau q^*_\tau\right] - B\right)\,\frac{\mathbb{1}(r_\tau = \underline{r}_\tau)}{\mathbb{P}(\mathcal{E})}.$$

The retailers' hedging strategies satisfy $G^*_\tau = w^*_\tau\, q^*_\tau - B - D^*_\tau$. *Finally, the expected payoffs of the producer and the retailers satisfy*

$$\Pi_{\mathrm{P}} = \frac{N\,\mathbb{E}\Big[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\Big]}{4\,(N+1)(1+\underline{r}_\tau)} \quad \textit{and} \quad \Pi_{\mathrm{R}} = \underline{r}_\tau\, B + \frac{\mathbb{E}\Big[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\Big]}{4\,(N+1)^2}.$$

We now make some observations regarding Proposition 3. We see that $\overline{B}_\tau$ is the budget B satisfying $\mathbb{E}\left[w^*_\tau q^*_\tau\right] = B$ where w^*_τ and q^*_τ are the optimal unconstrained price menus and ordering quantities, respectively, from regions I and II. Hence any value of $B \geq \overline{B}_\tau$ is such that the optimal unconstrained price menu and ordering quantities are achievable. Similarly we see that $\overline{\overline{B}}_\tau = \sup_{\omega\in\Omega} w^*_\tau q^*_\tau$, so no hedging (or borrowing) is required for budgets $B \geq \overline{\overline{B}}_\tau$. In Region II we have more than enough budget to achieve the unconstrained optimal and hence there are infinitely many optimal values of G^*_τ in this region (unless $B = \overline{B}_\tau$ in which case there is a uniquely optimal G^*_τ). In regions III and IV G^*_τ is uniquely determined by the budget constraint given the optimal w^*_τ, q^*_τ and D^*_τ. D^*_τ is only non-zero in Region IV and (as was the case in the Cournot equilibrium of Proposition 2) all borrowing in that region takes place on the event $\mathcal{E}$ where the borrowing rate is minimal. On that event $\mathcal{E}$ we borrow just enough to satisfy the budget constraint *in expectation* given w^*_τ and q^*_τ. The optimal hedging gain G^*_τ is then determined to ensure the budget constraint is satisfied almost surely, i.e. state-by-state.

In summary, the results of Proposition 3 suggest there is a pecking order to the hedging and debt components of the financial markets. While retailers rely on hedging for all values of $B < \overline{\overline{B}}_\tau$, they only use costly debt financing when their budget is very limited, i.e. when $B < \underline{B}_\tau$, and when they do borrow they do so in states where the interest rate is at its lowest possible level.

4.1 Our Illustrative Example Continued

We illustrate these features of the Cournot-Stackelberg equilibrium by revisiting our example from Section 3.1. We also compare this (price) *unconstrained* equilibrium to the Cournot-Stackelberg equilibrium where the producer is constrained to use a constant wholesale price menu[10]. We will refer to this latter equilibrium as the *static* equilibrium. The reason for considering the static equilibrium is twofold: first, it serves as a simple and natural benchmark to assess the impact that an adaptive wholesale price menu has on the supply chain and on the players' payoffs. Second, a constant wholesale price menu might be more practical to implement in many applications.

Let the tuple $(\Pi^*_{\mathrm{P}}(B), \Pi^*_{\mathrm{R}}(B), w^*(B), q^*(B))$ denote the unconstrained equilibrium from Proposition 3 as a function of B. Similarly, let $(\Pi^{\mathrm{S}}_{\mathrm{P}}(B), \Pi^{\mathrm{S}}_{\mathrm{R}}(B), w^{\mathrm{S}}(B), q^{\mathrm{S}}(B))$ denote the static equilibrium. Each of the four panels in Figure 4 depicts one of these quantities for each of the two equilibria (unconstrained vs. static) as a function of B. Since $w^*(B)$, $q^*(B)$ and $q^{\mathrm{S}}(B)$ are random variables, the figure shows their expected values $\mathbb{E}[w^*(B)]$, $\mathbb{E}[q^*(B)]$ and $\mathbb{E}[q^{\mathrm{S}}(B)]$, respectively. The four panels are divided into four shaded regions corresponding to the four regions defined in Proposition 3.

Beyond the comments we made immediately after the statement of Proposition 3, several further observations are in order. First, the equilibrium outcome $(\Pi_{\mathrm{P}}(B), \Pi_{\mathrm{R}}(B), w^*(B), q^*(B))$ is locally

[10]To compute this constant wholesale price equilibrium, we solve the producer's problem in equations (16)-(17) imposing the additional constraint that w_τ is a constant.

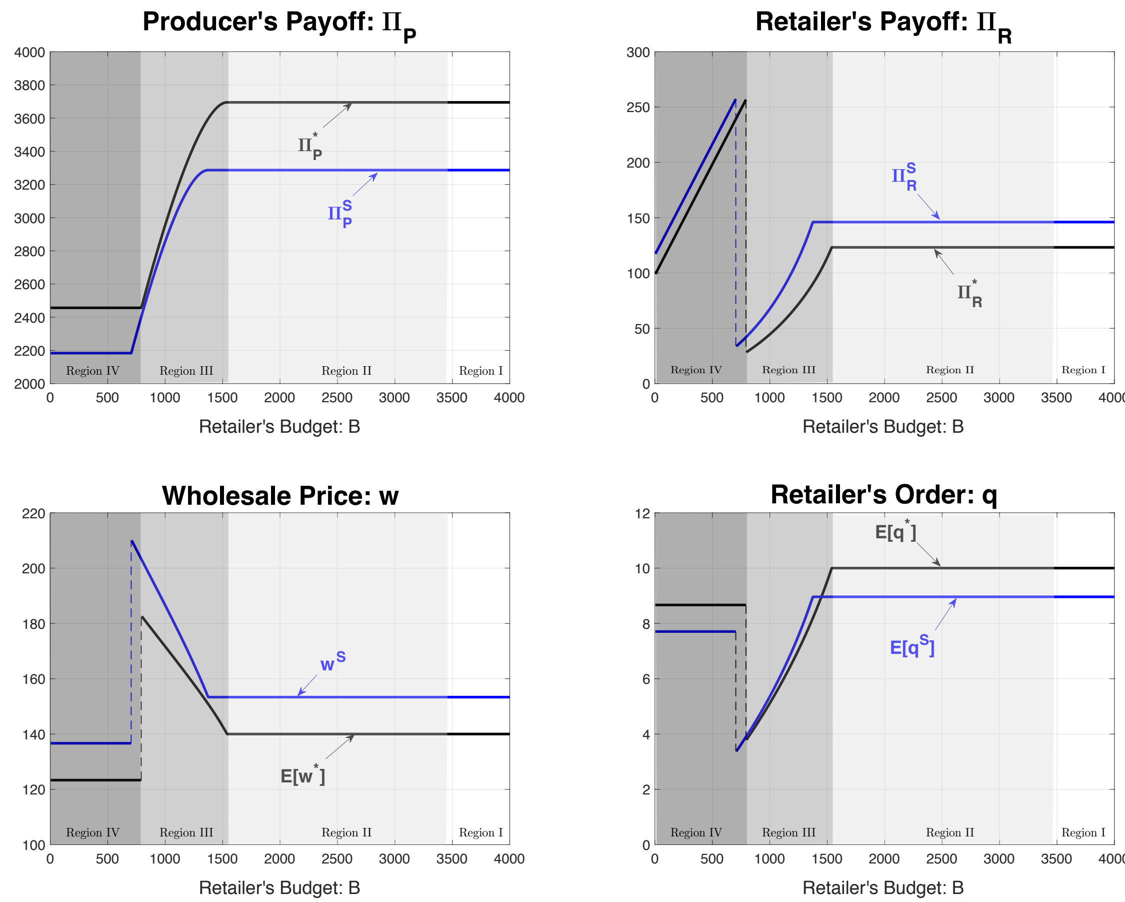

Figure 4: Cournot-Stackelberg equilibria $(\Pi_{\mathrm{P}}(B), \Pi_{\mathrm{R}}(B), \mathbb{E}[w^*(B)], \mathbb{E}[q^*(B)])$ and $(\Pi_{\mathrm{P}}^{\mathrm{S}}(B), \Pi_{\mathrm{R}}^{\mathrm{S}}(B), w^{\mathrm{S}}(B), \mathbb{E}[q^{\mathrm{S}}(B)])$ as a function of the retailer's budget B. In this example, the values of the threshold budgets identified in Proposition 3 are $\underline{B}_\tau = 793$, $\overline{B}_\tau = 1,539$ and $\overline{\overline{B}}_\tau = 3,483$. DATA: $N = 5$, $A_\tau \sim \mathrm{Uniform}[100, 300]$, $c_\tau = 80$ and $\underline{r}_\tau = 20\%$.

constant as a function of B in Regions I, II and IV with the exception of the retailers' payoff $\Pi_{\mathrm{R}}(B)$ in region IV where it increases linearly with B at a rate $\underline{r}_\tau$. We also note how the retailers' initial budget affects the producer's and retailers' payoffs differently in equilibrium. While the producer's expected payoff is non-decreasing in B, the retailers' expected payoff is not. For low values of B, $\Pi_{\mathrm{R}}(B)$ is increasing in B (region IV), it has a discontinuous drop at $B = \underline{B}_\tau$ and is non-decreasing for $B \geq \underline{B}_\tau$. At the point of discontinuity[11], the wholesale price jumps upward producing a downward jump on the retailers' ordering quantities.

It is also interesting to compare the unconstrained and static equilibria. We see the producer is uniformly better off in the unconstrained equilibrium whereas the retailers are uniformly worse off there. Of course the producer must be better off in the unconstrained equilibrium since he is always free in that setting to adopt a constant price menu. Yet it is interesting to see how much better off he is in the unconstrained equilibrium especially for very small and also for larger values of B. Nonetheless, from a qualitative perspective the two equilibria are very similar. This has implications for more realistic settings where the producer may not be completely free (for

[11] We discuss this debt-induced discontinuity further in Appendix C.

practical, regulatory or perhaps other reasons) to impose a completely unconstrained price menu.

5 The Value of Postponement

Now that we have characterized the (unconstrained) Cournot-Stackelberg equilibrium in Proposition 3, we turn to the question of measuring the value created by the interplay between production postponement and access to the financial markets. Recall there are two concrete means by which the financial markets add value to the operations of the system: (i) as a direct mechanism for mitigating the retailers' budget constraints through costly borrowing and dynamic trading and (ii) as a source of public information upon which the ordering quantities and wholesale prices are contingent. To isolate the effects of these two sources of value we will study them separately.

5.1 The Value of Financial Trading

To measure the value created by financial trading we will compare the output of the full Cournot-Stackelberg equilibrium of Proposition 3 to the Cournot-Stackelberg equilibrium of a system in which retailers have no access to the financial market. As before, we denote by Π_{P} and Π_{R} the payoffs of the producer and a retailer, respectively, for the Cournot-Stackelberg equilibrium of Proposition 3. We also define $\Pi_{\mathrm{P}}^{\mathrm{NT}}$ and $\Pi_{\mathrm{R}}^{\mathrm{NT}}$ to be the expected payoffs of the producer and a retailer, respectively, for the Cournot-Stackelberg equilibrium when the retailers do not engage in any form of financial trading (so the script 'NT' denotes 'No Trading'), i.e. no hedging and no borrowing. The Cournot-Stackelberg equilibrium for this case can be found in Appendix B.1. We define the value of financial trading for the producer and retailers as follows:

$$\text{Value of Financial Trading:} \qquad V_k^{\mathrm{FT}} := \Pi_k - \Pi_k^{\mathrm{NT}}, \quad k = \mathrm{P}, \mathrm{R}.$$

Since financial trading involves both hedging and borrowing, it is instructive to decompose the value of financial trading in terms of these two forms of financing. To this end, we let $\Pi_{\mathrm{P}}^{\mathrm{B}}$ and $\Pi_{\mathrm{R}}^{\mathrm{B}}$ denote the expected payoffs of the producer and a retailer, respectively, for the Cournot-Stackelberg equilibrium in which the retailers use the financial markets exclusively for borrowing and cannot hedge. The solution to the Cournot-Stackelberg equilibrium with only borrowing can be found in Appendix B.3. We have the following decomposition:

$$V_k^{\mathrm{FT}} = \underbrace{\Pi_k - \Pi_k^{\mathrm{B}}}_{\text{Value of Hedging}} + \underbrace{\Pi_k^{\mathrm{B}} - \Pi_k^{\mathrm{NT}}}_{\text{Value of Borrowing}} =: V_{k|\mathrm{H}}^{\mathrm{FT}} + V_{k|\mathrm{B}}^{\mathrm{FT}}, \quad k = \mathrm{P}, \mathrm{R}. \tag{21}$$

Equation (21) decomposes the value of financial trading into two quantities, namely the value of hedging $V_{k|\mathrm{H}}^{\mathrm{FT}} := \Pi_k - \Pi_k^{\mathrm{B}}$ and the value of borrowing $V_{k|\mathrm{B}}^{\mathrm{FT}} := \Pi_k^{\mathrm{B}} - \Pi_k^{\mathrm{NT}}$. To be clear, we could have used an alternative decomposition. In particular, let $\Pi_{\mathrm{P}}^{\mathrm{H}}$ and $\Pi_{\mathrm{R}}^{\mathrm{H}}$ denote the payoffs of the producer and a retailer, respectively, for the Cournot-Stackelberg equilibrium in which the retailers use the financial markets exclusively for hedging and have no access to borrowing. The solution to this equilibrium may be found in Appendix B.2. We could then write

$$V_k^{\mathrm{FT}} = \underbrace{\Pi_k - \Pi_k^{\mathrm{H}}}_{\text{Value of Borrowing}} + \underbrace{\Pi_k^{\mathrm{H}} - \Pi_k^{\mathrm{NT}}}_{\text{Value of Hedging}}, \quad k = \mathrm{P}, \mathrm{R} \tag{22}$$

and this would yield alternative measures for the value of hedging and borrowing, respectively. Rather than analyze both decompositions, however, we will focus instead on the decomposition given by (21) as this seems more natural in the following sense. Traditionally many companies have engaged in borrowing to deal with budget constraints but the idea of using dynamic hedging to (partly) circumvent a budget constraint is much more recent and, to the best of our knowledge, not nearly so widespread in practice. It therefore seems very unlikely that a firm would engage in hedging without also engaging in borrowing but the converse is not true, i.e. there may be (perhaps many) firms that borrow but don't hedge. These observations support the decomposition in (21) rather than (22) and so we shall focus on the former here.

Figure 5 depicts the values of trading, hedging and borrowing (as defined by (21)) for the producer and retailers as a function of the retailer's budget B. As before, each panel is divided into the four shaded regions defined in Proposition 3.

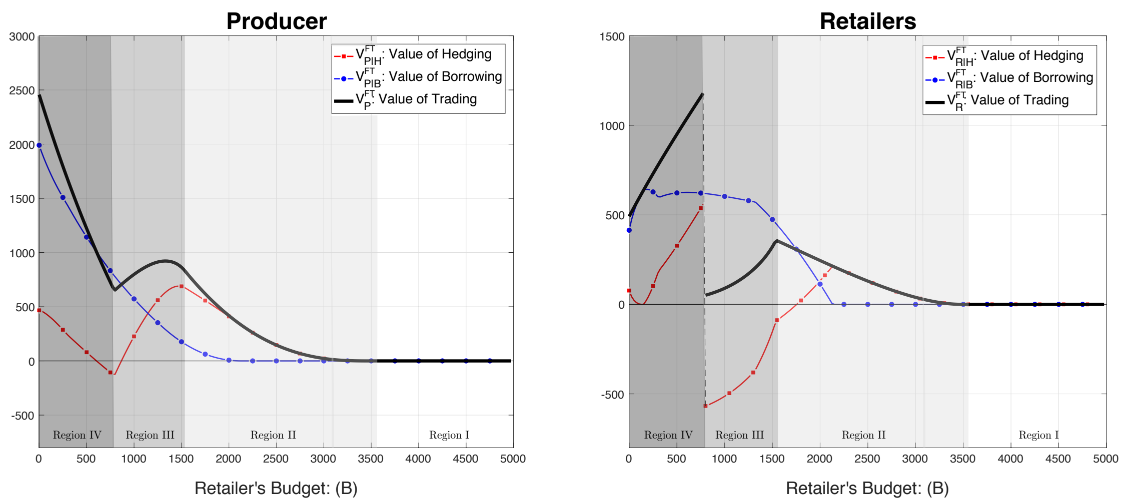

Figure 5: Agents' expected welfare as a function of the retailer's budget B. DATA: $N = 5$, $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $\underline{r}_\tau = 20\%$.

As we can see in this example, the value of financial trading is non-negative for both the producer and the retailers for all values of B. That is, retailers' access to the financial market improves the equilibrium expected payoffs for themselves as well as the producer. However, the same is not necessarily true when we decompose the value of trading into the values of hedging and borrowing. As we we see from the figure, the value of borrowing is also non-negative for all values of B but the value of hedging can be negative for some intermediate values of B. In other words, having access to borrowing always improves the expected payoffs of the agents while financial hedging can have a negative impact on their payoffs.

5.2 The Value of Production Postponement

We now consider the value of production postponement by measuring the difference between the payoffs that an agent obtains from the full Cournot-Stackelberg equilibrium of Proposition 3 and the payoffs that the agent obtains if procurement orders must be placed at time $t = 0$. We can recover[12] the Cournot-Stackelberg equilibrium for this case by setting $\tau = 0$ in Proposition 3 so that the random variable A_τ is replaced by the constant $A_0 = \mathbb{E}[A]$. As in Section 3.2 we again let $c_\tau = \alpha_\tau\, c_0$ and $(1 + \underline{r}_\tau) = \beta_\tau\,(1 + r_0)$, where α_τ and β_τ are monotonic non-decreasing and non-increasing functions of τ, respectively, and where $\beta_\tau \le 1 \le \alpha_\tau$.

We measure the value of postponement from the perspectives of five relevant '*parties*': the producer (P), the retailers (R), the supply chain (SC), the end consumers (C) and a social planner (SP). The payoff of the supply chain is the sum of the payoffs of the producer and retailers, that is, $\Pi_{\mathrm{SC}} := \Pi_{\mathrm{P}} + N \times \Pi_{\mathrm{R}}$. On the other hand, we measure the payoffs of the consumers as their expected surplus, namely, $\Pi_{\mathrm{C}} = \mathbb{E}[Q_\tau^2]/2$. Finally, for the social planner we will use the aggregate welfare $\Pi_{\mathrm{SP}} := \Pi_{\mathrm{SC}} + \Pi_{\mathrm{C}}$. For each of these five agents we define the value of postponement by

$$\textsf{Value of Postponement:}\quad V_k^{\mathrm{P}} = \Pi_k - \Pi_{k|0} \qquad k = \mathrm{P,\ R,\ SC,\ C,\ SP},$$

where $\Pi_{k|0}$ is the payoff of agent k when $\tau = 0$, i.e., there is no production postponement.

We will continue to work with the same running example that we have used previously. Specifically, we will assume $N = 5$, $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $\underline{r}_\tau = 0.2$ and $\mathcal{E} = \{A_\tau \ge 250\}$. We will also assume that $\alpha_\tau = 1.1$ (so that $c_0 = 72.7$) and $\beta_\tau = 0.9$ (so that $r_0 = 33.3\%$).

Figure 6 depicts the expected value of postponement for each of the five parties as a function of the retailer's budget B. As before, each panel is divided into the four shaded regions defined in Proposition 3. As we can see from the figure, the value of postponement varies significantly with B and across agents. For instance, in this example the value of postponement is positive for the producer, the supply chain and the social planner regardless of the value of B. For the retailers and the consumers, however, production postponement can be positive or negative depending on the value of B. It is also interesting to see that the value of postponing is not monotonic in B for any of the agents.

Our previous discussion reveals that, in general, the question of whether production postponement offers a positive value for a particular agent needs to be answered on a case-by-case basis using the results in Proposition 3. We can, however, derive some concrete insights in some special cases. The next proposition provides sufficient conditions under which all parties prefer production postponement for sufficiently large values of B or in the extreme case in which $B \downarrow 0$.

Proposition 4 *All five parties (producer, retailers, supply chain, end consumers and social planner) prefer production postponement in either of the following two cases:*

(a) $B \ge \overline{B}_\tau$ and $\alpha_\tau \ge 1$ satisfies

$$\mathbb{E}\big[\big((A_\tau - \alpha_\tau\, c_0)^+\big)^2\big] \ge \big((A_0 - c_0)^+\big)^2. \tag{23}$$

(b) In the limit as $B \downarrow 0$ if α_τ and β_τ satisfy $0 < \beta_\tau \le \alpha_\tau\, \beta_\tau < 1$.

Proposition 4 sheds some light on some of the key trade-offs involved with production postponement. Part (a) assumes the retailers are not budget constrained (possibly due to the ability to

[12] Alternatively we can also recover this equilibrium by setting $\tau = 0$ in the statements of Propositions 8 (Cournot equilibrium) and 9 (Cournot-Stackelberg equilibrium) of Appendix B.3 where we consider the borrowing but no hedging setting.

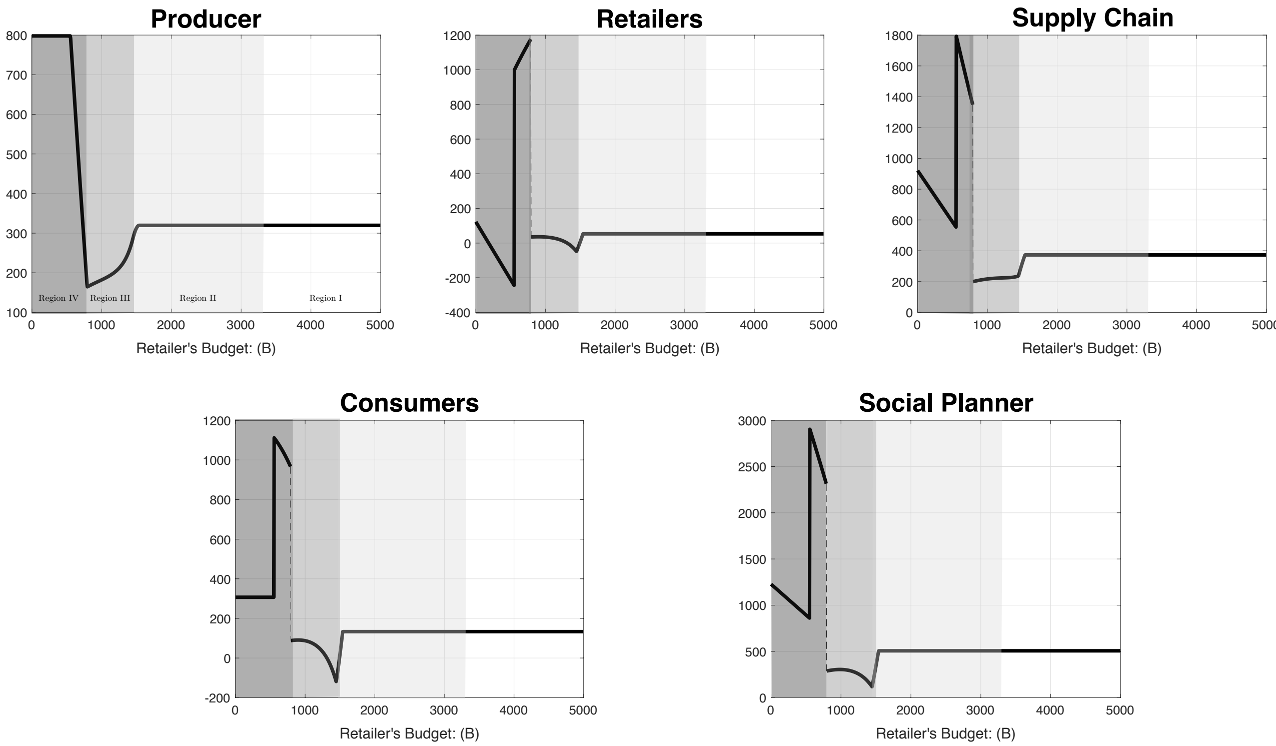

Figure 6: Value of postponement (V_k^{P}) for the different agents as a function of the retailer's budget B. DATA: $N = 5$, $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $\underline{r}_\tau = 20\%$, $\alpha_\tau = 1.1$ and $\beta_\tau = 0.9$.

hedge) and this is the standard 'deep pocket' assumption that is made in most of the operations management literature on production postponement. In this case, the benefit of postponement is measured by the trade-off between the incremental cost of delaying production decisions and the benefits of postponing these decisions to gather better demand information. The general conclusion in this setting is that production postponement is more valuable when future demand is more volatile; see Swaminathan and Tayur (1998) and Kisperska-Moron and Swierczek (2011). In our model, this intuition is captured by condition (23) but now instead of using the volatility of the market size A we use a quantity that approximates the volatility of $A_\tau := \mathbb{E}_\tau[A]$. To understand condition (23), note if there is no information in the financial markets regarding the market outcome A, then A_τ would be a constant and equal to $A_0 = \mathbb{E}[A]$. It then follows that condition (23) will not be satisfied for any $\alpha_\tau > 1$ and so production postponement is not uniformly valuable for all five agents in this case. (It might still be of value to a strict subset of the agents but that would depend on the specific value of α_τ and the other model parameters.)

Part (b) of Proposition 4 also reveals that there is another factor influencing the value of postponement when retailers are budget constrained. In this case, by postponing ordering decisions and their corresponding payments, retailers are able to take full advantage of the financial markets to (i) reallocate their limited budget across different states of nature using hedging and (ii) decrease the financial cost of debt as they will only borrow on those future states where interest rates are the lowest. By looking at the extreme case in which $B \downarrow 0$, we can see that production postponement can be beneficial for all parties even if the incremental cost of delaying procurement α_τ is arbitrarily large as long as the reduction in the cost of debt β_τ is sufficiently large, namely, as long as the condition $\alpha_\tau \, \beta_\tau < 1$ is satisfied. We believe this factor has received very little attention in the production postponement literature. We also note that when $\alpha_\tau = 1$ (so production costs do not increase with τ) then all agents will be better off postponing for sufficiently small B as long as $\beta_\tau < 1$ which, as argued earlier, is a very reasonable assumption. Finally, as with part (a) the conditions stated in part (b) are sufficient to guarantee *all* agents will prefer postponement. For individual agents, we can obtain looser conditions under which postponement is preferred. In the proof of part (b) of the proposition, for example, we show the producer prefers postponement more generally for $B \leq \underline{B}_\tau$ (with α_τ and β_τ still required to satisfy $0 < \beta_\tau \leq \alpha_\tau \, \beta_\tau < 1$).

6 Competition in the Retailers' Market

We now investigate how the Cournot-Stackelberg equilibrium is affected by the degree of competition in the retailers' market, specifically by the number of retailers N. To this end, we find it convenient to make explicit the dependence of the different components of the model on N. For example, we will write $B(N)$ for the budget of an individual retailer or $w_\tau(N)$ for the wholesale price menu in a market with N retailers. In order to isolate the impact of N on the equilibrium outcome, we will assume the cumulative budget B_{C} remains constant as we vary N, i.e. we assume $B_{\mathrm{C}} = N \, B(N)$ does not vary with N. Thus, our sensitivity analysis and results in this section are exclusively driven by the degree of competition in the retailers' market and not by an increase in the cumulative budget available as the number of retailers increases.

Recall from Proposition 3 that the solution of the Cournot-Stackelberg equilibrium can be divided into four cases depending on the value of the initial budget $B(N)$ of each retailer. These four cases are depicted in Figure 7 and correspond to the four regions in the statement of Proposition 3. The threshold functions $\overline{\overline{B}}_{\mathrm{C}}(N)$, $\overline{B}_{\mathrm{C}}(N)$ and $\underline{B}_{\mathrm{C}}(N)$ that separate the four regions in Figure 7 are given by[13]

$$\overline{\overline{B}}_{\mathrm{C}}(N) := \frac{N \, \sup\left\{(A_\tau^2 - c_\tau^2)^+\right\}}{4(N+1)}, \quad \overline{B}_{\mathrm{C}}(N) := \frac{N \, \mathbb{E}\left[(A_\tau^2 - c_\tau^2)^+\right]}{4(N+1)} \quad \text{and} \quad \underline{B}_{\mathrm{C}}(N) = N \, \underline{B}_\tau(N),$$

where $\underline{B}_\tau(N)$ is defined in (18). A couple of observations are in order. First we see that these boundary functions are increasing in N and converge asymptotically to constants that do not depend on N. Second, for a fixed value of B_{C} it is possible to transition from one region to another as N increases. For example, when $B_{\mathrm{C}} \approx .75$ in Figure 7, we are in Region II for $N \leq 4$ but then cross into Region III where we remain for all values of $N > 5$.

This transitioning from one region to another can cause interesting behavior in the equilibrium when viewed as a function of N. We discuss this below but first we state a result that follows directly[14] from Proposition 3 and that reveals the behavior of the Cournot-Stackelberg equilibrium

[13] These threshold values are simply the corresponding thresholds defined immediately before the statement of Proposition 3 multiplied by N.

[14] We omit the proof as it follows from some simple algebraic manipulations of the various optimal quantities from the four cases of Proposition 3.

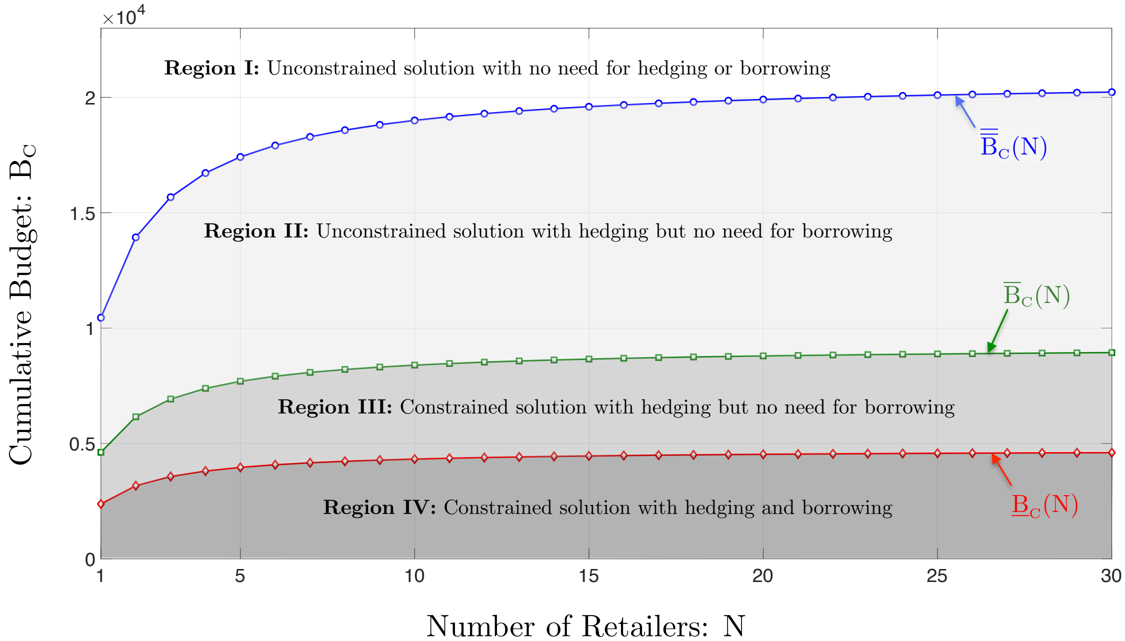

Figure 7: Characteristics of the Cournot-Stackelberg equilibrium as defined in Proposition 3 in B_{C} vs. N space. DATA: $A_\tau \sim \mathrm{Uniform}[100, 300]$, $c_\tau = 80$ and $\underline{r}_\tau = 20\%$.

as a function of N is similar in Regions I, II, and IV. In fact, Regions I and II have the same Cournot-Stackelberg equilibrium in terms of wholesale price, output quantities and firms' payoffs. (Recall the only difference is that in Region II the retailers need to hedge their budget constraints while in Region I they do not.)

Proposition 5 *Within the interior of regions I, II, and IV the Cournot-Stackelberg equilibrium satisfies:*

(a) *The wholesale prices $w_\tau(N)$ are constant independent of N.*

(b) *The total market output $Q_\tau(N) = N\, q_\tau(N)$ is increasing in N.*

(c) *The producer's payoff $\Pi_{\mathrm{P}}(N)$ is increasing in N. The retailers' cumulative payoff $N\,\Pi_{\mathrm{R}}(N)$ is decreasing in N.*

(d) *The consumers' surplus $\Pi_{\mathrm{C}}(N) := \mathbb{E}[(Q_\tau)^2(N)]/2$ and the aggregate social welfare $\Pi_{\mathrm{SP}}(N)$ are increasing in N.*

Within the interior of regions I and II we also have:

(e) The expected payoff to the overall supply chain $\Pi_{\mathrm{SC}}(N)$ is increasing in N.

It is interesting that regions I and II correspond to the case in which the retailers' budget is sufficiently high that the unconstrained optimal solution is attainable in equilibrium while region IV corresponds to the case in which the retailers' budget is sufficiently low that borrowing is required. Thus, according to part (d) in Proposition 5, if the cumulative retailers' budget B_C is sufficiently high or sufficiently low then higher levels of competition in the retailers' market lead to more efficient Cournot-Stackelberg equilibrium outcomes despite the fact that the retailers themselves are worse off.

In contrast, this positive effect that retailers' competition has on market efficiency does not extend to the case in which B_C is in the intermediate level of region III. Indeed Figure 8 depicts the consumers' surplus $\Pi_C(N)$ (left panel) and aggregate social welfare $\Pi_{SP}(N)$ (right panel) for a problem instance in which the equilibrium switches[15] from Region II to Region III as N increases. For comparison, each panel also includes the case in which the retailers have no access to the financial market for either hedging or borrowing.

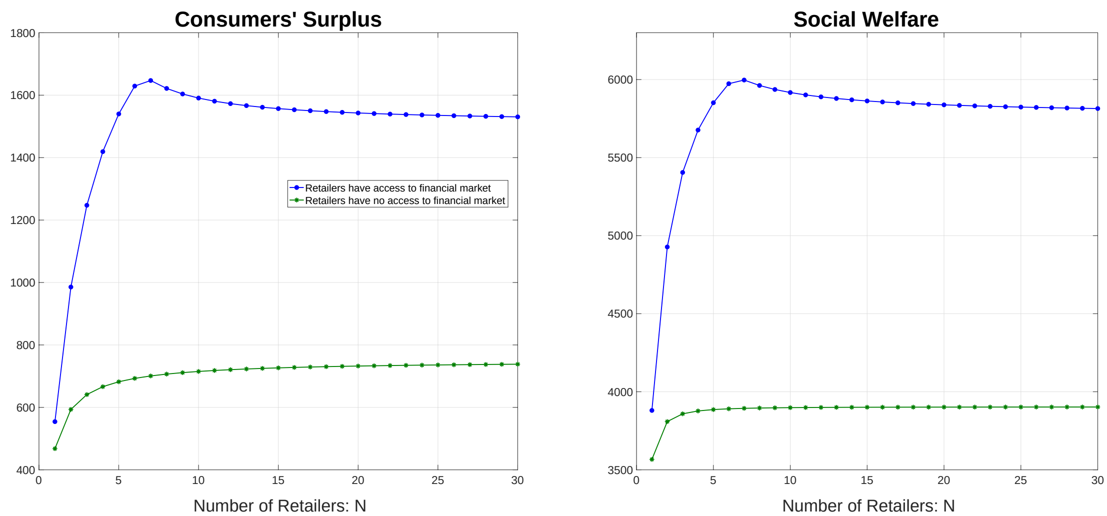

Figure 8: Consumers' expected surplus (left panel) and social welfare (right panel) as a function of N for the cases in which the retailers have access (blue curve) and don't have access (green curve) to the financial market. DATA: $A_\tau \sim \text{Uniform}[100, 300]$, $c_\tau = 80$, $\underline{r}_\tau = 20\%$ and $B_C = 8,000 < \lim_{N\to\infty} \overline{B}_C(N)$.

First, we note the end consumers and society as a whole are better off when retailers have access to the financial market for a given value of N. Of particular interest is that when the retailers have access to the financial markets, the consumers' surplus and social welfare are maximized at some finite value $N = N^*$. This is in direct contrast with the result in Proposition 5 that shows that when the retailers' cumulative budget is sufficiently small or large, the consumers' surplus and social welfare are maximized when $N = \infty$, i.e. when retailers' competition is most intense. However, when the cumulative budget is at an intermediate value, there is a socially optimal number of retailers ($N^* = 7$ in this example) and excessive competition in the retailers' market can end up hurting consumers, the supply chain and society as a whole. In contrast, when the retailers do not have access to the financial markets, consumers' surplus and social welfare increase as N gets large. That is, in the absence of hedging and borrowing, more competition in the retailers' market is better for both consumers and society as a whole.

[15] In this numerical example the switch took place at the peak where $N = 7$. Note that part (d) of Proposition 5, however, only guarantees that in general the switch from region II to region III must occur on or before any such peak.

To formalize the previous discussion, we define the threshold

$$N^*(B_{\mathrm{C}}) := \left\lfloor \frac{4\,B_{\mathrm{C}}}{(\mathbb{E}[(A_\tau^2 - c_\tau^2)^+] - 4\,B_{\mathrm{C}})^+} \right\rfloor \tag{24}$$

which corresponds to the inverse mapping of $\overline{B}_{\mathrm{C}}(N)$. In this definition, we allow for $N^*(B_{\mathrm{C}}) = \infty$ if $\mathbb{E}[(A_\tau^2 - c^2)^+] \leq 4B_{\mathrm{C}}$. (The positive part in the denominator of (24) is used to ensure that the threshold is nonnegative.) We also note that $\overline{B}_{\mathrm{C}}(N) = \mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/8$ when $N = 1$ and $\lim_{N\to\infty} \overline{B}_{\mathrm{C}}(N) = \mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/4$. Therefore if B_{C} lies between these two values then we are guaranteed to cross from region II into region III at a value of $N^*(B_{\mathrm{C}})$ satisfying $0 < N^*(B_{\mathrm{C}}) < \infty$; see Figure 7. We have the following Proposition.

Proposition 6 *Suppose that B_{C} is such that $\mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/8 < B_{\mathrm{C}} < \mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/4$ and that retailers have full access to the financial markets. Then in equilibrium:*

a) *The expected wholesale price $\mathbb{E}[w_\tau(N)]$ does not depend on N for $N \leq N^*(B_{\mathrm{C}})$ and is strictly increasing in N for $N > N^*(B_{\mathrm{C}})$.*

b) *Suppose that A_τ admits a smooth density $f \in \mathcal{C}^2[0,\infty)$ such that $\lim_{x\to\infty} x^2 f(x) = 0$. The expected market output $\mathbb{E}[Q_\tau(N)] = \mathbb{E}[N\,q_\tau(N)]$ is increasing in N for $N \leq N^*(B_{\mathrm{C}})$ and decreasing in N for $N > N^*(B_{\mathrm{C}})$.*

c) *The expected profits of the producer increases with N while the aggregate expected profits of the retailers decrease with N.*

d) *The consumers' surplus $\Pi_{\mathrm{C}}(N)$, the cumulative profits of the supply chain $\Pi_{\mathrm{CH}}(N) = \Pi_{\mathrm{P}}(N) + N\,\Pi_{\mathrm{R}}(N)$, and the social welfare $\Pi_{\mathrm{SP}}(N)$ are all strictly increasing in N for $N \leq N^*(B_{\mathrm{C}})$.*

An immediate corollary of part (c) is that consolidations in the retail market always benefit the retailers and hurt the producer. We also emphasize that the results in Proposition 6 hold only in expectation and assume the retailers have access to the financial markets. On a state-by-state basis the producer will not necessarily be better off as the number of retailers increases. For example, there will be some outcomes of A_τ where the ordering quantity is zero when there are multiple competing retailers and the ordering quantity is strictly positive when there is a single (merged) retailer. The producer will earn zero profits on such paths under the competing retailer model, but will earn strictly positive profits under the merged retailer model.

Part (d) of Proposition 6 implies that, independently of the measure of welfare that we might adopt, i.e. the consumers', the supply chain's or society's, $N^*(B_{\mathrm{C}})$ is a lower bound on the minimum number of retailers that should be operating in the system from a welfare standpoint when the cumulative budget B_{C} is at an intermediate value.

7 Conclusions and Further Research

We have studied the impact of financial postponement on the operations of a stylized supply chain where N identical retailers and a single producer compete in a Cournot-Stackelberg game. The retailers purchase a single product from the producer and afterwards sell it in the retail market at a stochastic clearance price that depends in part on the terminal value of a tradeable financial asset. We therefore considered a variation of the traditional wholesale price contract where at $t = 0$ the producer offers a menu of wholesale prices to the retailers, one for each realization of the financial asset price at time τ. The retailers then commit to purchasing at time τ a variable number of units, with the specific quantity depending on the time τ asset price. The retailers are budget constrained but can hedge and borrow in the financial markets to partly mitigate this. After completely characterizing the Cournot and Cournot-Stackelberg equilibrium we compared them to various equilibrium benchmarks where hedging and / or borrowing are not available to the retailers. We showed there is a pecking order to the hedging and debt components of the financial markets whereby hedging is used at low and intermediate budget levels while debt is only used at low budget levels. We identify conditions under which the producer, retailers, consumers and a central planner are all better off by postponing production. We also studied the impact of retail competition on the equilibrium. We showed that higher levels of competition in the retailers' market increase supply chain efficiency, consumers' surplus and social welfare when the retailers' budgets are either high or low. For intermediate budget levels, however, it's possible that too much retailer competition can have a detrimental effect on these measures.

There are several possible directions for future research. One direction is to consider alternative contracts such as 2-part tariffs for coordinating the supply chain. Of course we would still like to have these contracts be contingent upon the outcome of the financial markets. Another direction would be the development of a model where the retailers compete via strategic complements rather than via strategic substitutes which is the case we consider here. On the technical side we could also consider the problem of finding an optimal postponement time τ. This problem could be formulated as a constrained optimal stopping problem and solved numerically. Finally, we would like to characterize the Cournot-Stackelberg equilibrium in the general case where the retailers don't have identical budgets.

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A Appendix: Proofs

PROOF OF LEMMA 1 : Note that the i^{th} retailer's problem in (9) to (11) decouples by the state $\omega \in \Omega$ so we only need consider the problem for a fixed state. The Lagrangian for the i^{th} retailer's problem is then given by

$$L(q_{i\tau}, D_{i\tau}) = (A_\tau - (q_{i\tau} + Q_{i\tau-}) - w_\tau)\, q_{i\tau} - r_\tau\, D_{i\tau} + \lambda(B_{i\tau} + D_{i\tau} - w_\tau\, q_{i\tau}) + \mu D_{i\tau} + \gamma q_{i\tau}$$

where λ, μ and γ are non-negative Lagrange multipliers for the constraints in (10) and (11). (We omit the dependence of these multipliers on i.) The first order conditions are

$$\begin{aligned} A_\tau - 2q_{i\tau} - Q_{i\tau-} - (1+\lambda)w_\tau + \gamma &= 0 \\ -r + \lambda + \mu &= 0 \end{aligned}$$

together with the inequality constraints. Complimentary slackness and the first-order conditions then yield

$$\begin{aligned} D^*_{i\tau} &= (w_\tau \, q^*_{i\tau} - B_{i\tau})^+ \\ q^*_{i\tau} &= \frac{(A_\tau - Q_{i\tau-} - (1+\lambda)w_\tau)^+}{2} \end{aligned} \tag{A-1}$$

with $0 \le \lambda \le r_\tau$. We now only need to identify the value of λ in (A-1).

Suppose first that $B_{i\tau} < 0$. In that case it follows (since $q^*_{i\tau} \ge 0$ and we assume $w_\tau \ge 0$) that $D^*_{i\tau} > 0$ which implies $\lambda = r_\tau$ and we are done. So now assume $B_{i\tau} \ge 0$ and let α be the solution to the equation

$$B_{i\tau} = w_\tau \frac{(A_\tau - Q_{i\tau-} - (1+\alpha)w_\tau)^+}{2}. \tag{A-2}$$

If $\alpha < 0$ then the non budget-constrained solution $q_{i\tau} = (A_\tau - Q_{i\tau-} - w_\tau)^+/2$ satisfies $w_\tau \, q_{i\tau} \le B_{i\tau}$ and so $\lambda = 0$. If $0 \le \alpha \le r_\tau$, then we can take $\lambda = \alpha$ in (A-1) and the budget constraint is satisfied with no need to borrow. Finally if $\alpha > r_\tau$ then (since $0 \le \lambda \le r_\tau$) we see from (A-1) that borrowing is required to satisfy the budget constraint and so $\mu = 0$ and $\lambda = r_\tau$. In summary, we have $\lambda_{i\tau} = \min\left\{r_\tau, \alpha^+_{i\tau}\right\}$ where we now explicitly recognize the dependence on i. $\square$

Proof of Proposition 1 : We begin by subtracting $q^*_{i\tau}/2$ from both sides of (A-1). A little algebra then yields

$$q^*_{i\tau} = (A_\tau - Q_\tau - (1+\lambda_{i\tau})w_\tau)^+$$

and summing this over i yields

$$Q_\tau = \sum_{i=1}^{N} (A_\tau - Q_\tau - (1+\lambda_{i\tau})w_\tau)^+ . \tag{A-3}$$

The statement of the proposition now follows from the final part of the proof of Lemma 1 which identifies the three possible values that each summand on the r.h.s. of (A-3) can take. $\square$

Lemma 2 below characterizes retailer i's best-response $q^*_{i\tau}(w_\tau, Q_{i\tau-})$ for a given wholesale price menu w_τ and the cumulative orders $Q_{i\tau-}$ of the other retailers. This best response is the solution of (4)-(6) taking w_τ and $Q_{i\tau-}$ as given. Before proceeding we recall $\underline{r}_\tau := \inf_{\omega\in\Omega}\{r_\tau\}$ is the lowest possible interest rate at which the retailers can borrow. For fixed values of w_τ and $Q_{\tau-}$ and for some real scalar x we also define

$$\mathcal{B}(w_\tau, Q_{\tau-}, x) := \mathbb{E}\left[w_\tau \, \frac{(A_\tau - Q_{\tau-} - w_\tau\,(1+x))^+}{2}\right] \tag{A-4}$$

and we let $\alpha(w_\tau, Q_{\tau-}, B)$ be the unique solution of the equation $B = \mathcal{B}(w_\tau, Q_{\tau-}, \alpha)$.

Lemma 2 (Retailer's Best Response) *Let w_τ be the producer's price menu and $Q_{i\tau-}$ the cumulative orders of all retailers excluding retailer i. Then the i^{th} retailer's optimal order is*

$$q_{i\tau}^* = \frac{(A_\tau - Q_{i\tau-} - w_\tau\,(1+\lambda_i))^+}{2}$$

where $\lambda_i := \max\{0, \min\{\underline{r}_\tau, \alpha(w_\tau, Q_{i\tau-}, B)\}\}$. More specifically, we have the following three cases:

- Case 1: *If $B > \mathcal{B}(w_\tau, Q_{i\tau-}, 0)$ then $\lambda_i = 0$, the retailer uses no debt, i.e., $D_{i\tau}^* = 0$ and can use infinitely many hedging strategies that can implement the optimal ordering quantity $q_{i\tau}^*$. One particular choice is*

$$G_{i\tau}^* = (w_\tau\, q_{i\tau}^* - B)\cdot \begin{cases} \delta_\tau & \text{if } \omega \in \mathcal{X}_\tau \\ 1 & \text{if } \omega \in \mathcal{X}_\tau^c \end{cases} \tag{A-5}$$

where $\delta_\tau := \dfrac{\int_{\mathcal{X}_\tau^c} [w_\tau\, q_{i\tau}^* - B]\, \mathrm{d}\mathbb{Q}}{\int_{\mathcal{X}_\tau} [B - w_\tau\, q_{i\tau}^*]\, \mathrm{d}\mathbb{Q}}$, $\mathcal{X}_\tau := \{\omega \in \Omega \;:\; B \geq w_\tau\, q_{i\tau}^*\}$ *and* $\mathcal{X}_\tau^c := \Omega - \mathcal{X}_\tau$.

- Case 2: *If $\mathcal{B}(w_\tau, Q_{i\tau-}, \underline{r}_\tau) < B < \mathcal{B}(w_\tau, Q_{i\tau-}, 0)$ then $\lambda_i = \alpha(w_\tau, Q_{i\tau-}, B)$, retailer i does not raise any debt, i.e. $D_{i\tau}^* = 0$, and uses the hedging strategy $G_{i\tau}^* = w_\tau\, q_{i\tau}^* - B$.*

- Case 3: *If $B \leq \mathcal{B}(w_\tau, Q_{i\tau-}, \underline{r}_\tau)$ then $\lambda_i = \underline{r}_\tau$ and the retailer only raises debt on the event $\mathcal{E} := \{r_\tau = \underline{r}_\tau\}$. Moreover there are infinitely many optimal borrowing strategies D_τ^* that the retailer can use with the only requirement being that $\mathbb{E}[D_{i\tau}^*] = \mathbb{E}[w_\tau\, q_{i\tau}^*] - B$. One specific choice that borrows uniformly on $\mathcal{E}$ is given by*

$$D_{i\tau}^* = \left(\frac{\mathbb{E}[w_\tau\, q_{i\tau}^*] - B}{\mathbb{P}(\mathcal{E})}\right) \mathbb{1}(r_\tau = \underline{r}_\tau).$$

Finally, the retailer's optimal hedging strategy satisfies $G_{i\tau}^* = w_\tau\, q_{i\tau}^* - B - D_{i\tau}^*$.

Proof of Lemma 2 : For notational convenience, we will drop the dependence of quantities on the index i. For example, we will write q_τ or λ instead of $q_{i\tau}$ or λ_i.

Let β_τ and λ denote Lagrange multipliers for the budget and hedging constraints, respectively. Note that while λ is a deterministic scalar, β_τ is stochastic since the budget constraint must be satisfied pathwise. Similarly, we define η_τ and θ_τ to be Lagrange multipliers for the non-negativity constraints $q_\tau \geq 0$ and $D_\tau \geq 0$, respectively. The first-order optimality conditions for the relaxed version of the problem are given by

$$\begin{aligned} &q_\tau = \frac{A_\tau - Q_{\tau-} - w_\tau\,(1+\beta_\tau) + \eta_\tau}{2}, \quad \beta_\tau = \lambda, \quad \theta_\tau + \beta_\tau = r_\tau, \quad \beta_\tau\,(B + G_\tau + D_\tau - w_\tau\, q_\tau) = 0, \\ &\eta_\tau\, q_\tau = 0, \quad \theta_\tau\, D_\tau = 0, \quad \mathbb{E}\,[G_\tau] = 0, \quad B + G_\tau + D_\tau \geq w_\tau\, q_\tau, \quad q_\tau, D_\tau, \beta_\tau, \eta_\tau, \theta_\tau \geq 0. \end{aligned} \tag{A-6}$$

From the complementary slackness condition $\eta_\tau\, q_\tau = 0$ and the non-negativity of the production level q_τ and multiplier η_τ, at optimality we must have

$$q_\tau^* = \frac{(A_\tau - Q_{\tau-} - w_\tau\,(1+\lambda))^+}{2}. \tag{A-7}$$

In solving the first-order optimality conditions in (A-6), we identify different cases depending on the value of λ. First, note that λ cannot be negative since at optimality $\lambda = \beta_\tau \geq 0$. In addition, λ cannot be greater than $\underline{r}_\tau$ since $\theta_\tau = r_\tau - \beta_\tau = r_\tau - \lambda \geq 0$. Hence, at optimality we must have $\lambda \in [0, \underline{r}_\tau]$. In what follows we distinguish three cases depending on whether λ is a boundary or an interior value in this interval.

- Case 1: Suppose at optimality $\lambda = 0$. Then (A-7) yields

$$q_\tau^* = \overline{q}_\tau := \frac{(A_\tau - Q_{\tau-} - w_\tau)^+}{2}.$$

The first-order optimality conditions also then imply $\beta_\tau = 0$, $\theta_\tau = r_\tau > 0$ (since $r_\tau > 0$ by assumption) and so $D_\tau^* = 0$ by complementary slackness. For $\overline{q}_\tau$ to be feasible it must satisfy the budget constraint for some G_τ satisfying the hedging constraint $\mathbb{E}[G_\tau] = 0$. Taking expectations across the budget constraint we see that B must satisfy

$$B \geq \mathbb{E}[w_\tau \overline{q}_\tau] = \mathcal{B}(w_\tau, Q_{i\tau-}, 0).$$

Finally, under this condition on the retailer's budget, one can show that there are infinitely many optimal hedging strategies G_τ^* that satisfy $\mathbb{E}[G_\tau^*] = 0$ and can sustain the budget constraint for $q_\tau = \overline{q}_\tau$. One particular choice is given by (A-5).

In summary, when $B \geq \mathcal{B}(w_\tau, Q_{i\tau-}, 0)$ the retailer can implement the first-best production level $q_\tau^* = \overline{q}_\tau$ without using any debt, i.e. with $D_\tau^* = 0$.

- Case 2: Suppose at optimality $0 < \lambda < \underline{r}_\tau$. Then $\theta_\tau = r_\tau - \lambda > 0$ since $\lambda < \underline{r}_\tau \leq r_\tau$ which implies $D_\tau^* = 0$ by the complementary slackness condition $\theta_\tau\, D_\tau = 0$. In addition, $\beta_\tau = \lambda > 0$ which again by complementary slackness implies $B + G_\tau^* = w_\tau\, q_\tau^*$. By substituting for q_τ^* using (A-7), then taking expectations and recalling $\mathbb{E}[G_\tau^*] = 0$, we obtain

$$B = \mathbb{E}\left[w_\tau \, \frac{(A_\tau - Q_{\tau-} - w_\tau\,(1+\lambda))^+}{2}\right] = \mathcal{B}(w_\tau, Q_{i\tau-}, \lambda). \tag{A-8}$$

This is an equation in λ which admits a unique solution in $(0, \underline{r}_\tau)$ if $\mathcal{B}(w_\tau, Q_{i\tau-}, \underline{r}_\tau) < B < \mathcal{B}(w_\tau, Q_{i\tau-}, 0)$.

- Case 3: Suppose at optimality $\lambda = \underline{r}_\tau$. It follows from (A-7) that

$$q_\tau^* = \frac{(A_\tau - Q_{\tau-} - w_\tau\,(1+\underline{r}_\tau))^+}{2}.$$

Furthermore, since $\lambda > 0$, we see the budget constraint is satisfied with equality, i.e. $B + G_\tau^* + D_\tau^* - w_\tau\, q_\tau^* = 0$. Taking expectations, we see that an optimal borrowing strategy must satisfy

$$\mathbb{E}[D_\tau^*] = \mathbb{E}[w_\tau\, q_\tau^*] - B. \tag{A-9}$$

In addition, by complementary slackness we have $D_\tau^* = 0$ on $\mathcal{E}^c$ where $\mathcal{E}$ is defined in the statement of the lemma. It follows that the retailer will only borrow in those states belonging to $\mathcal{E}$. There are infinitely many borrowing strategies that satisfy (A-9) and $D_\tau^* = 0$ in $\mathcal{E}^c$. One special case is the strategy that borrows uniformly in $\mathcal{E}$. That is,

$$D^*_\tau = \left(\frac{\mathbb{E}[w_\tau\, q^*_\tau] - B}{\mathbb{P}(\mathcal{E})}\right) \mathbb{1}(r_\tau = \underline{r}_\tau).$$

Finally, an optimal hedging strategy is given by $G^*_\tau = w_\tau\, q^*_\tau - B - D^*_\tau$.

This completes the proof of the lemma. □

<u>PROOF OF PROPOSITION 2:</u> (A-7) in the proof of Lemma 2 reveals that retailer i's optimal ordering quantity $q^*_{i\tau}$ is a function of a parameter $\lambda_i \in [0, \underline{r}_\tau]$, which corresponds to the Lagrange multiplier of the hedging constraint $\mathbb{E}[G_\tau] = 0$. This relationship suggests that we can characterize a Cournot equilibrium in the space

$$\Lambda := \Big\{(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N_+ \,:\, \lambda_i \in [0, \underline{r}_\tau] \quad \text{for all } i = 1, 2, \ldots, N\Big\}.$$

Indeed, given a vector of multipliers $(\lambda_1, \lambda_2, \ldots, \lambda_N)$, Lemma 2 implies the corresponding inventory level of retailer i satisfies

$$q_{i\tau} = \frac{(A_\tau - Q_\tau + q_{i\tau} - w_\tau\,(1+\lambda_i))^+}{2} \quad \text{for all } i = 1, 2, \ldots, N$$

which is equivalent to

$$q_{i\tau} = (A_\tau - Q_\tau - w_\tau\,(1+\lambda_i))^+ \quad \text{for all } i = 1, 2, \ldots, N.$$

It follows that the aggregate inventory level Q_τ solves the fixed-point condition

$$Q_\tau = \sum_{i=1}^{N} (A_\tau - Q_\tau - w_\tau\,(1+\lambda_i))^+ \tag{A-10}$$

which depends uniquely on the vector of Lagrange multipliers $(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Thus, to complete the proof we only need to show that in equilibrium the Lagrange multiplier λ_i for retailer i is independent of i (i.e., $\lambda_i = \lambda_j$ for $i \neq j$). Suppose, by contradiction, that there exist two retailers i and j such that $\lambda_i < \lambda_j$ in equilibrium. It follows that $q_{i\tau} \geq q_{j\tau}$ and so $Q_{i\tau-} \leq Q_{j\tau-}$. Hence, from the definition of $\alpha(w_\tau, Q_{\tau-}, B)$ we can show that $\alpha(w_\tau, Q_{i\tau-}, B) \geq \alpha(w_\tau, Q_{j\tau-}, B)$. But we also have the identity $\lambda_k := \max\{0, \min\{\underline{r}_\tau, \alpha(w_\tau, Q_{k\tau-}, B)\}\}$ for all $k = 1, 2, \ldots, N$, which implies $\lambda_i \geq \lambda_j$. This contradicts our assumption $\lambda_i < \lambda_j$ and we conclude that $\lambda_i = \lambda_j$. It therefore follows from (A-10) that in equilibrium $Q_\tau = N(A_\tau - Q_\tau - w_\tau\,(1+\lambda))^+$ where λ is the common equilibrium Lagrange multiplier and from this the expression for q^*_τ in (14) follows. We can then substitute $(N-1)q^*_\tau$ for $Q_{\tau-}$ in (A-4) and see that the definition of $\alpha^*(B)$ given in the statement of the Proposition coincides with the definition of α given just before the statement of Lemma 2. The rest of the proof then follows from the results of Lemma 2.

Note that the previous argument also shows the Cournot game admits a unique symmetric equilibrium in terms of the ordering quantities. □

Lemma 3 *$\underline{B}$ as defined by (18) and (19) is well-defined and unique.*

Proof of Lemma 3 : We first simplify the notation and define $X := 4\,(N+1)\,\underline{B}$. Note that ϕ^* in (19) is trivially a function of X. The uniqueness of $\underline{B}$ follows if we can show that the r.h.s. of (18) is increasing in X or equivalently, that

$$\frac{\mathrm{d}}{\mathrm{d}X}\, 2\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(X)\,c_\tau\right)^+\right] < 1. \tag{A-11}$$

From (19) it follows that if X is sufficiently large then $\phi^*(X) \equiv 1$ and so (A-11) is satisfied trivially. Suppose now that X is such that $\phi^*(X) > 1$. Then the inequality in (19) is satisfied with equality and can be re-arranged as

$$2\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(X)\,c_\tau\right)^+\right] + \mathbb{E}\left[(A_\tau - \phi^*(X)\,c_\tau)^+\,(A_\tau + (\phi^*(X) - 2)\,c_\tau)\right] = X.$$

So to prove (A-11) we need to show that $\frac{\mathrm{d}}{\mathrm{d}X}\,\mathbb{E}\left[(A_\tau - \phi^*(X)\,c_\tau)^+\,(A_\tau + (\phi^*(X) - 2)\,c_\tau)\right] > 0$ or equivalently that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}\phi}\,\mathbb{E}\left[(A_\tau - \phi\,c_\tau)^+\,(A_\tau + (\phi - 2)\,c_\tau)\right]\Big|_{\phi=\phi^*(X)}}_{(A)} \times \underbrace{\frac{\mathrm{d}\phi^*(X)}{\mathrm{d}X}}_{(B)} > 0. \tag{A-12}$$

Now, it is not hard to see that for those values of X for which $\phi^*(X) > 1$ the derivative in (B) is strictly negative. In additional, the function $H(\phi) = (A_\tau - \phi\,c_\tau)^+\,(A_\tau + (\phi - 2)\,c_\tau)$ is strictly decreasing in ϕ in the region $\phi > 1$. Thus, the derivative in (A) is also strictly negative and so the product of the two derivatives in (A-12) is positive as desired. $\square$

Proof of Proposition 3: We must solve[16] the following optimization problem:

$$\Pi_{\mathrm{P}} = \max_{w_\tau \geq 0}\ \frac{N}{N+1}\ \mathbb{E}\Big[(w_\tau - c_\tau)\ (A_\tau - w_\tau\,(1 + \lambda^*(w_\tau)))^+\Big] \tag{A-13}$$

$$\text{s.t.}\quad \lambda^*(w_\tau) = \max\Big\{0, \min\{\underline{r}_\tau, \alpha^*(w_\tau)\}\Big\},\ \text{where } \alpha^*(w_\tau) \text{ solves } \mathbb{E}\left[w_\tau\,\frac{(A_\tau - w_\tau\,(1+\alpha^*))^+}{(N+1)}\right] = B.$$

To find a solution we partition the space of feasible contracts $\mathcal{W} := \{w_\tau \geq 0\}$ into three regions $\mathcal{W}_1 := \{w_\tau \geq 0 : \alpha^*(w_\tau) \leq 0\}$, $\mathcal{W}_2 := \{w_\tau \geq 0 : 0 < \alpha^*(w_\tau) \leq \underline{r}_\tau\}$ and $\mathcal{W}_3 := \{w_\tau \geq 0 : \alpha^*(w_\tau) > \underline{r}_\tau\}$, and then define the corresponding optimization subproblems

$$\Pi_{\mathrm{P}}^i = \max_{w_\tau \in \mathcal{W}_i}\ \frac{N}{N+1}\ \mathbb{E}\Big[(w_\tau - c_\tau)\ (A_\tau - w_\tau\,(1 + \lambda^*(w_\tau)))^+\Big] \tag{A-14}$$

for $i = 1, 2, 3$. We analyze the three subproblems in turn:

- Subproblem 1: In this case, $\alpha^*(w_\tau) \leq 0$ so that $\lambda^*(w_\tau) = 0$ since $\underline{r}_\tau > 0$. The optimization problem in (A-14) reduces to

$$\Pi_{\mathrm{P}}^1 = \max_{w_\tau \geq 0}\ \frac{N}{N+1}\ \mathbb{E}\Big[(w_\tau - c_\tau)\ (A_\tau - w_\tau)^+\Big] \quad \text{subject to} \quad \mathbb{E}\left[w_\tau\,\frac{(A_\tau - w_\tau)^+}{(N+1)}\right] \leq B.$$

[16]In Proposition 2 we wrote $\alpha^*(B)$ as a function of B as w_τ was held fixed there. But here w_τ is a decision variable and so now we write α^* and therefore λ^* as a function of w_τ.

By relaxing the constraint and solving the KKT conditions we find an optimal solution is given by

$$w_\tau^1 = \frac{A_\tau + \phi^* c_\tau}{2} \qquad \text{and} \qquad \Pi_{\mathrm{P}}^1 = \frac{N}{4\,(N+1)}\mathbb{E}\Big[(A_\tau + (\phi^* - 2)\, c_\tau)\,(A_\tau - \phi^*\, c_\tau)^+\Big],$$

where

$$\phi^* = \min\Big\{\phi \geq 1 \,:\, \mathbb{E}\Big[(A_\tau^2 - \phi^2\, c_\tau^2)^+\Big] \leq 4\,(N+1)\, B\Big\}.$$

- Subproblem 2: In this case $0 < \lambda^*(w_\tau) = \alpha^*(w_\tau) \leq \underline{r}_\tau$ and (A-14) reduces to

$$\Pi_{\mathrm{P}}^2 = N\,B - \min_{\substack{w_\tau \geq 0 \\ 0<\lambda^* \leq \underline{r}_\tau}} \;\frac{N}{N+1}\;\mathbb{E}\Big[c_\tau\;(A_\tau - w_\tau\,(1+\lambda^*))^+\Big] \quad \text{s.t.} \quad \mathbb{E}\left[w_\tau\;\frac{(A_\tau - w_\tau\,(1+\lambda^*))^+}{(N+1)}\right] = B.$$

By inspection, we can see that the solution to this problem cannot have $\lambda^* > 0$. This can be shown by contradiction. Indeed, suppose that there exists an optimal solution (w_τ^*, λ^*) such that $\lambda^* > 0$. Let $\epsilon > 0$ be small enough so that $\lambda^* - \epsilon > 0$ and define $\delta > 0$ such that

$$H(\delta) := \mathbb{E}\left[(w_\tau^* + \delta)\,\frac{(A_\tau - (w_\tau^* + \delta)\,(1 + \lambda^* - \epsilon))^+}{(N+1)}\right] = B.$$

The existence of such a $\delta > 0$ follows the fact the function $H(\delta)$ is continuous in δ and satisfies $H(0) > B$ and $\lim_{\delta\to\infty} H(\delta) = 0$. It follows that $(w_\tau^* + \delta, \lambda^* - \epsilon)$ is feasible for Subproblem 2. This feasibility also implies that

$$\mathbb{E}\Big[c_\tau\;(A_\tau - w_\tau^*\,(1+\lambda^*))^+\Big] > \mathbb{E}\Big[c_\tau\;(A_\tau - (w_\tau^* + \delta)\,(1 + \lambda^* - \epsilon))^+\Big].$$

- Subproblem 3: In this case $\alpha^*(w_\tau) > \underline{r}_\tau$ and so $\lambda^*(w_\tau) = \underline{r}_\tau$. The problem in (A-14) therefore reduces to

$$\Pi_{\mathrm{P}}^3 = \max_{w_\tau \geq 0}\;\frac{N}{N+1}\;\mathbb{E}\Big[(w_\tau - c_\tau)\;(A_\tau - w_\tau(1+\underline{r}_\tau))^+\Big] \quad \text{subject to} \quad \mathbb{E}\left[w_\tau\;\frac{(A_\tau - w_\tau(1+\underline{r}_\tau))^+}{(N+1)}\right] > B.$$

Given the strict inequality on the constraint, the feasible region is open and the optimization problem can only admit an interior solution. By inspection, this interior solution is given by

$$w_\tau^3 = \frac{A_\tau + (1+\underline{r}_\tau)\,c_\tau}{2\,(1+\underline{r}_\tau)}$$

which satisfies the strict inequality constraint if

$$B < \frac{\mathbb{E}\Big[(A_\tau^2 - (1+\underline{r}_\tau)^2\,c_\tau^2)^+\Big]}{4\,(N+1)\,(1+\underline{r}_\tau)} := \mathcal{B}_3. \tag{A-15}$$

If this condition is not satisfied then the optimization problem in Case 3 does not admit a solution in the sense that a solution would be a boundary condition that would violate the strict inequality of the constraint. On the other hand, if the condition is satisfied, the producer's payoff in Case 3 is equal to

$$\Pi_{\mathrm{P}}^3 = \frac{N}{4\,(N+1)\,(1+\underline{r}_\tau)}\;\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right].$$

Based on these three subproblems, we identify two possible cases depending on the value of B:

1. Large Budget: Suppose that $B \geq \mathcal{B}_3$, i.e., condition (A-15) is not satisfied. Then, only Subproblem 1 has an optimal solution and we conclude that the producer optimal wholesale price and payoff are given by

$$w_\tau^* = \frac{A_\tau + \phi^*(B)\, c_\tau}{2} \qquad \text{and} \qquad \Pi_{\mathrm{P}}^* = \frac{N}{4\,(N+1)} \mathbb{E}\Big[(A_\tau + (\phi^*(B) - 2)\, c_\tau)\,(A_\tau - \phi^*(B)\, c_\tau)^+\Big],$$

where

$$\phi^*(B) = \min\Big\{\phi \geq 1 \,:\, \mathbb{E}\Big[(A_\tau^2 - \phi^2\, c_\tau^2)^+\Big] \leq 4\,(N+1)\,B\Big\}.$$

2. Small Budget: Suppose that $B < \mathcal{B}_3$, i.e. condition (A-15) is satisfied. In this case, Subproblems 1 and 3 have an optimal solution. Therefore, the producer will select the solution that gives the higher expected payoff. Thus, we need to decide under what conditions the producer's payoff in Subproblem 1 exceeds the payoffs in Subproblem 3, i.e. when is $\Pi_{\mathrm{P}}^1 \geq \Pi_{\mathrm{P}}^3$. We claim this is the case as long as $B \geq \underline{B}$, where $\underline{B}$ is the value of the retailers' initial budget for which $\Pi_{\mathrm{P}}^1 = \Pi_{\mathrm{P}}^3$. Therefore $\underline{B}$ solves the equation

$$\frac{1}{1+\underline{r}_\tau}\,\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] = \mathbb{E}\left[\Big(A_\tau^2 - (\phi^*(\underline{B})\,c_\tau)^2\Big)^+ - 2\,c_\tau\,\Big(A_\tau - \phi^*(\underline{B})\,c_\tau\Big)^+\right]. \tag{A-16}$$

To prove this claim, we first show the producer's payoff from Subproblem 1, i.e. $\Pi_{\mathrm{P}}^1(B)$, is a non-decreasing function of B. To see this, recall the producer's payoff in Subproblem 1 is equal to

$$\Pi_{\mathrm{P}}^1(B) = \frac{N}{N+1}\mathbb{E}\Big[(w_\tau^1(B) - c_\tau)\,(A_\tau - w_\tau^1(B))^+\Big] \qquad \text{where} \quad w_\tau^1(B) = \frac{A_\tau + \phi^*(B)\,c_\tau}{2}.$$

But the function $h(w) := (w - c_\tau)\,(A_\tau - w)^+$ has a maximum at $w_\tau^* = (A_\tau + c_\tau)/2$. Hence, since the function $\phi^*(B)$ is non-increasing in B and by definition satisfies $\phi^*(B) \geq 1$, we have that $w_\tau^1 \geq w_\tau^*$ and is also non-increasing in B. We conclude that $\Pi_{\mathrm{P}}^1(B)$ is non-decreasing in B.

It is also easy to see that $\Pi_{\mathrm{P}}^1(0) = 0 \leq \Pi_{\mathrm{P}}^3$. Also, for $B \geq \overline{B} = \mathbb{E}\Big[(A_\tau^2 - c_\tau^2)^+\Big]/(4\,(N+1))$ we have that $\phi^*(B) = 1$ and so

$$\Pi_{\mathrm{P}}^1(B) = \frac{N}{4\,(N+1)}\mathbb{E}\Big[\Big((A_\tau - c_\tau)^+\Big)^2\Big] \geq \frac{N}{4\,(N+1)\,(1+\underline{r}_\tau)}\,\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] = \Pi_{\mathrm{P}}^3.$$

It now follows from the monotonicity of $\Pi_{\mathrm{P}}^1(B)$ and the independence of $\Pi_{\mathrm{P}}^3(B)$ on B that there exists a unique value $\underline{B}$ such that $\Pi_{\mathrm{P}}^1(\underline{B}) = \Pi_{\mathrm{P}}^3$ and that $\Pi_{\mathrm{P}}^1(\underline{B}) \geq \Pi_{\mathrm{P}}^3$ for all $B \geq \underline{B}$.

To complete the proof, we need to argue that for $B < \underline{B}$, the optimal solution is given by Subproblem 3 and for $B \geq \underline{B}$, the optimal solution is given by Subproblem 1. By our previous discussion, this is the same as showing that $\underline{B} \leq \mathcal{B}_3$. To show this inequality, note first that $\underline{B} < \overline{B}$ because $\Pi_{\mathrm{P}}^1(B)$ is non-decreasing in B and

$$\begin{aligned}\Pi_{\mathrm{P}}^1(\underline{B}) = \Pi_{\mathrm{P}}^3 \quad &= \quad \frac{N}{4\,(N+1)\,(1+\underline{r}_\tau)}\,\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] \\ &< \quad \frac{N}{4\,(N+1)}\,\mathbb{E}\left[\Big((A_\tau - c_\tau)^+\Big)^2\right] = \Pi_{\mathrm{P}}^1(\overline{B})\end{aligned}$$

where the strict inequality is due to the fact that $\underline{r}_\tau > 0$.

Since $\underline{B} < \overline{B}$ it follows that $\phi^*(\underline{B}) > 1$ and so $\mathbb{E}\Big[(A_\tau^2 - \phi^2(\underline{B})\, c_\tau^2)^+\Big] = 4\,(N+1)\,\underline{B}$. Also, from equation (A-16), we get that

$$\underline{B} = \frac{1}{4\,(N+1)\,(1+\underline{r}_\tau)}\left(\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] + 2\,(1+\underline{r}_\tau)\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(\underline{B})\,c_\tau\right)^+\right]\right).$$

Therefore, the condition $\underline{B} \leq \mathcal{B}_3$ is equivalent to

$$\left(\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] + 2\,(1+\underline{r}_\tau)\,\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(\underline{B})\,c_\tau\right)^+\right]\right) \leq \mathbb{E}\Big[(A_\tau^2 - (1+\underline{r}_\tau)^2\,c_\tau^2)^+\Big]$$

which after some manipulations leads to the condition

$$\mathbb{E}\left[c_\tau\left(A_\tau - \phi^*(\underline{B})\,c_\tau\right)^+\right] \leq \mathbb{E}\left[c_\tau\left(A_\tau - (1+\underline{r}_\tau)\,c_\tau\right)^+\right]. \tag{A-17}$$

Proving (A-17) amounts to showing that $\phi^*(\underline{B}) \geq 1 + \underline{r}_\tau$ and towards this end we define the function

$$F(\phi) := \mathbb{E}\left[\left(A_\tau^2 - (\phi\,c_\tau)^2\right)^+ - 2\,c_\tau\left(A_\tau - \phi\,c_\tau\right)^+\right]$$

which corresponds to the right-hand side of (A-16) viewed as a function of ϕ. Now it's easy to check that $F(\phi)$ is decreasing in the domain $\phi \geq 1$. We also have

$$\begin{aligned}
F(1+\underline{r}_\tau) &= \mathbb{E}\left[\left(A_\tau^2 - ((1+\underline{r}_\tau)\,c_\tau)^2\right)^+ - 2\,c_\tau\left(A_\tau - (1+\underline{r}_\tau)\,c_\tau\right)^+\right] \\
&= \mathbb{E}\left[\left(A_\tau - (1-\underline{r}_\tau)\,c_\tau\right)\left(A_\tau - (1+\underline{r}_\tau)\,c_\tau\right)^+\right] \geq \mathbb{E}\left[\left(A_\tau - (1+\underline{r}_\tau)\,c_\tau\right)\left(A_\tau - (1+\underline{r}_\tau)\,c_\tau\right)^+\right] \\
&\geq \frac{1}{1+\underline{r}_\tau}\,\mathbb{E}\left[\Big((A_\tau - (1+\underline{r}_\tau)\,c_\tau)^+\Big)^2\right] = \frac{4(N+1)}{N}\Pi_{\mathrm{P}}^3.
\end{aligned}$$

But by the definition of $\underline{B}$ in (A-16) we also have $F(\phi^*(\underline{B})) = 4(N+1)\Pi_{\mathrm{P}}^3/N$ then $\phi^*(\underline{B}) \geq 1 + \underline{r}_\tau$.

Finally, it is now just a matter of relating the results above to the four cases identified in the statement of the proposition. It's clear the first 3 cases correspond to budgets $B \geq \underline{B}$ while case 4 corresponds to budgets $B < \underline{B}$. Cases 1 and 2 correspond to budgets $B \geq \overline{B}$ where $\phi^*(B) = 1$ while case 3 corresponds to budgets satisfying $\underline{B} \leq B < \overline{B}$ which have $\phi^*(B) > 1$. We also note that cases 1 and 2 are almost identical with the only difference being whether or not any hedging is actually required. Case 1 applies when the budget constraint holds in each state without any need for hedging. This is when $B \geq \overline{\overline{B}}$ which is defined immediately preceding the proposition. When $\overline{B} \leq B < \overline{\overline{B}}$ some hedging is required and case 2 applies. We also note that $\lambda^*(B) = 0$ for $B \geq \underline{B}$ and $\lambda^*(B) = \underline{r}_\tau$ for $B < \underline{B}$. We can then use the results of Proposition 2 to obtain the optimal ordering quantities q_τ^*, hedging gains G_τ^* and debt D_τ^*. $\square$

The proof of Proposition 6 requires the following Lemma.

Lemma 4 *Let A be a nonnegative random variable that admits a smooth density $f \in \mathcal{C}^2[0,\infty)$ such that $\lim_{x\to\infty} x^2\, f(x) = 0$. Then, the function*

$$\bar{\mathcal{A}}(x) := \frac{\mathbb{E}[A\,(A-x)^+]}{\mathbb{E}[(A-x)^+]}$$

is increasing in x.

PROOF: First, let us rewrite $\bar{\mathcal{A}}(x)$ as follows:

$$\bar{\mathcal{A}}(x) = x + \frac{\mathbb{E}[(A-x)\,(A-x)^+]}{\mathbb{E}[(A-x)^+]} = x + \frac{\int_x^\infty (y-x)^2\, f(y)\,\mathrm{d}y}{\int_x^\infty (y-x)\, f(y)\,\mathrm{d}y} = x + \frac{\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y}{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}.$$

Taking derivatives w.r.t. x, and using the fact that $f \in \mathcal{C}^2[0,\infty)$ so that we can interchange differentiation and integration, we obtain

$$\frac{\mathrm{d}\bar{\mathcal{A}}(x)}{\mathrm{d}x} = 1 + \frac{\left(\int_0^\infty y^2\, f'(y+x)\,\mathrm{d}y\right)\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right) - \left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\left(\int_0^\infty y\, f'(y+x)\,\mathrm{d}y\right)}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}. \quad \text{(A-18)}$$

Now using integration by parts and the assumption on f that $\lim_{x\to\infty} x^2\, f(x) = 0$ we have

$$\int_0^\infty y^2\, f'(y+x)\,\mathrm{d}y = -2\int_0^\infty y\, f(y+x)\,\mathrm{d}y \qquad \text{and} \qquad \int_0^\infty y\, f'(y+x)\,\mathrm{d}y = -\int_0^\infty f(y+x)\,\mathrm{d}y.$$

Substituting these identities into (A-18) we obtain

$$\begin{aligned}
\frac{\mathrm{d}\bar{\mathcal{A}}(x)}{\mathrm{d}x} &= 1 + \frac{\left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\left(\int_0^\infty f(y+x)\,\mathrm{d}y\right) - 2\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2} \\
&= \frac{\left(\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y\right)\left(\int_0^\infty f(y+x)\,\mathrm{d}y\right) - \left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2}{\left(\int_0^\infty y\, f(y+x)\,\mathrm{d}y\right)^2} \\
&= \left(\frac{\int_0^\infty f(y+x)\,\mathrm{d}y}{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}\right)^2 \left[\frac{\int_0^\infty y^2\, f(y+x)\,\mathrm{d}y}{\int_0^\infty f(y+x)\,\mathrm{d}y} - \left(\frac{\int_0^\infty y\, f(y+x)\,\mathrm{d}y}{\int_0^\infty f(y+x)\,\mathrm{d}y}\right)^2\right] \geq 0.
\end{aligned}$$

The non-negativity of the previous expression follows by noting that the term inside the square brackets is the variance of a nonnegative random variable Y with density

$$f_Y(y) := \frac{f(y+x)}{\int_0^\infty f(z+x)\,\mathrm{d}z}.$$

The result now follows. □

PROOF OF PROPOSITION 4:

Part (a) follows from noticing that for $B \geq \overline{B}_\tau$ Proposition 3 shows that the producer's, retailer's and supply chain's payoffs as well as consumers' expected surplus and social welfare are all proportional to $\mathbb{E}\big[\big((A_\tau - c_\tau)^+\big)^2\big]$. Thus, using the fact that $c_\tau = \alpha_\tau\, c_0$ we conclude that all parties prefer postponement if $\mathbb{E}\big[\big((A_\tau -\, \alpha_\tau\, c_0)^+\big)^2\big] \geq \big((A_0 - c_0)^+\big)^2$.

To prove part (b) we again use the result in Proposition 3 for the case in which $B < \underline{B}_\tau$. In this case the producer's expected payoff with postponement is equal to

$$\Pi_{\mathrm{P}} = \frac{N\,\mathbb{E}\Big[\Big((A_\tau - (1+\underline{r}_\tau)\, c_\tau)^+\Big)^2\Big]}{4\,(N+1)(1+\underline{r}_\tau)} = \frac{N\,\mathbb{E}\Big[\Big((A_\tau - \alpha_\tau\, \beta_\tau\, (1+r_0)\, c_0)^+\Big)^2\Big]}{4\,(N+1)\beta_\tau\,(1+r_0)}$$

where the second equality uses the definitions $c_\tau = \alpha_\tau c_0$ and $(1+\underline{r}_\tau) = \beta_\tau(1+r_0)$. Using the assumption that $0 < \beta_\tau \leq \alpha_\tau \beta_\tau < 1$ we conclude that

$$\Pi_{\mathrm{P}} \geq \frac{N\,\mathbb{E}\Big[\Big((A_\tau - (1+r_0)\,c_0)^+\Big)^2\Big]}{4\,(N+1)(1+r_0)},$$

where the right-hand side is the producer's payoff without postponement.

A similar set of arguments can be used to show that the retailers' payoff, consumers' surplus and social welfare with postponement are also greater than than those without postponement under the same set of conditions. (Because of the term $\underline{r}_\tau B$ in the retailer's expected payoff we must let $B \downarrow 0$ in this case.) $\square$

<u>Proof of Proposition 6:</u> The discussion immediately preceding the statement of the proposition explains why $\mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/8 < B_{\mathrm{C}} < \mathbb{E}\big[(A_\tau^2 - c_\tau^2)^+\big]/4$ is required to guarantee that $1 < \bar{N}(B_{\mathrm{C}}) < \infty$ so that we cross from region II to region III at $\bar{N}(B_{\mathrm{C}}) < \infty$. The definition of $\bar{N}(B_{\mathrm{C}})$ then implies that for $N \leq \bar{N}(B_{\mathrm{C}})$ the Cournot-Stackelberg equilibrium lies in Region II while for $N > \bar{N}(B_{\mathrm{C}})$ the equilibrium lies in Region III in Figure 7. We now prove the four parts.

(a) Because of our initial observation above, according to Proposition 3 for $N \leq \bar{N}$ we have that

$$w_\tau(N) = \frac{A_\tau + c_\tau}{2} \qquad \text{and} \qquad Q_\tau(N) = \frac{N\ (A_\tau - c_\tau)^+}{2(N+1)}.$$

It follows trivially that $w_\tau(N)$ is constant and $Q_\tau(N)$ is increasing in N pathwise and therefore in expectation. In contrast, when $N > \bar{N}(B_{\mathrm{C}})$ we have

$$w_\tau(N) = \frac{A_\tau + \phi^*(B_{\mathrm{C}}/N)\,c_\tau}{2} \qquad \text{and} \qquad Q_\tau(N) = \frac{N}{N+1}\left(\frac{A_\tau - \phi^*(B_{\mathrm{C}}/N)\,c_\tau}{2}\right)^+$$

with $\phi^*(B)$ defined in (19). It is not hard to see that $\phi^*(B_{\mathrm{C}}/N)$ is strictly increasing in N if $N > \bar{N}(B_{\mathrm{C}})$ and so the wholesale price $w_\tau(N)$ is is also strictly increasing in N on each path and therefore in expectation if $N > \bar{N}(B_{\mathrm{C}})$.

(b) To show that the total expected output $\mathbb{E}[Q_\tau(N)]$ is decreasing in N when $N > \bar{N}(B_{\mathrm{C}})$ we note that $\phi^*(B_{\mathrm{C}}/N) > 1$ in this region and so (19) implies that $\mathbb{E}[Q_\tau(N)\,(A_\tau - \phi^*(B_{\mathrm{C}}/N)\,c_\tau)] = 2\,B_{\mathrm{C}}$ or equivalently

$$\phi^*(B_{\mathrm{C}}/N)\,c_\tau + \frac{\mathbb{E}[A_\tau\,Q_\tau(N)]}{\mathbb{E}[Q_\tau(N)]} = \frac{2\,B_{\mathrm{C}}}{\mathbb{E}[Q_\tau(N)]}.$$

From this identity, and the fact that $\phi^*(B_{\mathrm{C}}/N)$ is increasing in N for $N > \bar{N}(B_{\mathrm{C}})$, we will show that $\mathbb{E}[Q_\tau(N)]$ is decreasing in N by showing that the term $\mathbb{E}[A_\tau\,Q_\tau(N)]/\mathbb{E}[Q_\tau(N)]$ is increasing in N.

Towards this end, note that $\mathbb{E}[A_\tau\,Q_\tau(N)]/\mathbb{E}[Q_\tau(N)] = \bar{\mathcal{A}}(\phi^*(B_{\mathrm{C}}/N)\,c_\tau)$ where $\bar{\mathcal{A}}$ was defined in the statement of Lemma 4. Hence, since $\phi^*(B_{\mathrm{C}}/N)$ in increasing in N, the result of Lemma 4 completes the proof.

(c) From Case 2 in Proposition 3 we have that for $N \leq \bar{N}(B_{\mathrm{C}})$

$$\Pi_{\mathrm{P}}(N) = \frac{N}{4\,(N+1)}\,\mathbb{E}\left[\Big((A_\tau - c_\tau)^+\Big)^2\right] \qquad \text{and} \qquad \Pi_{\mathrm{R}}(N) = \frac{1}{4\,(N+1)^2}\,\mathbb{E}\left[\Big((A_\tau - c_\tau)^+\Big)^2\right].$$

It follows that $\Pi_{\mathrm{P}}(N)$ is increasing in N and $N\,\Pi_{\mathrm{R}}(N)$ is decreasing in N. Similarly, from Case 3 in Proposition 3, we have that for $N > \bar{N}(B_{\mathrm{C}})$

$$\Pi_{\mathrm{P}}(N) = B_{\mathrm{C}} - \frac{N}{2\,(N+1)}\,\mathbb{E}\left[c_\tau\left(A_\tau - (\phi^*(B_{\mathrm{C}}/N)\,c_\tau)\right)^+\right] \quad \text{and} \quad \Pi_{\mathrm{R}}(N) = \frac{\mathbb{E}\left[\left((A_\tau - \phi^*(B_{\mathrm{C}}/N)\,c_\tau)^+\right)^2\right]}{4\,(N+1)^2}.$$

Since $\phi^*(B_{\mathrm{C}}/N)$ is increasing in N for $N > \bar{N}(B_{\mathrm{C}})$, we have immediately that $N\,\Pi_{\mathrm{R}}(N)$ is decreasing in N. To show that $\Pi_{\mathrm{P}}(N)$ is increasing in N, note that $\Pi_{\mathrm{P}}(N)$ is equivalent to

$$\Pi_{\mathrm{P}}(N) = B_{\mathrm{C}} - c_\tau\,\mathbb{E}[Q_\mathcal{T}(N)],$$

but from the proof of part (b) we know that $\mathbb{E}[Q_\mathcal{T}(N)]$ is decreasing in N for $N > \bar{N}(B_{\mathrm{C}})$.

(d) Note again that for all $N \leq \bar{N}(B_{\mathrm{C}})$ we have that

$$Q_\mathcal{T}(N) = \frac{N\,(A_\tau - c)^+}{2(N+1)}$$

and so

$$\mathcal{C}(N) = \frac{N^2\,\mathbb{E}[\left((A_\tau - c_\tau)^+\right)^2]}{8\,(N+1)^2} \qquad \text{and} \qquad \Pi_{\mathrm{CH}}(N) = \Pi_{\mathrm{P}}(N) + N\,\Pi_{\mathrm{R}}(N) = \frac{N\,(N+2)}{4\,(N+1)^2}\mathbb{E}[\left((A_\tau - c_\tau)^+\right)^2].$$

We conclude that both $\mathcal{C}(N)$ and $\Pi_{\mathrm{CH}}(N)$ are increasing in N for $N \leq \bar{N}$. The proof is completed by noticing that social welfare $\mathcal{S}(N) = \mathcal{C}(N) + \Pi_{\mathrm{CH}}(N)$. $\square$

B Benchmarks

Here we derive the Cournot and Cournot-Stackelberg equilibria for the special cases of the model that we used as benchmarks in our numerical example from Sections 3.1, 4.1 and 5. In Appendix B.1 we consider the case in which the retailers have no access to the financial market and so they can neither hedge nor borrow. Then in Appendix B.2 we consider the case in which the retailers can use the financial markets to hedge their budget constraint but are unable to borrow. In Appendix B.3 we consider the reverse case in which the retailers can borrow but cannot hedge. Finally, in Appendix B.4 we consider the centralized version of the problem in which a single firm controls both production and retail operations. Due to space constraints we did not consider the centralized version of the problem in the main text but we include it here for the sake of completeness.

B.1 Decentralized Supply Chain With Neither Borrowing Nor Hedging

Our first benchmark is where the retailers have no access at all to the financial markets. It's worth noting that the financial markets still play a significant role in this problem formulation, however, since the retailers can condition their orders based on the value of $A_\tau = \mathbb{E}[A|\mathcal{F}_\tau]$. In this case, for a given wholesale price menu w_τ set by the producer, we can determine retailer i's order quantity by solving the best-response optimization problem:

$$\Pi_{\mathrm{R}_i}(w_\tau) = \max_{q_i \geq 0}\ \mathbb{E}\left[(A_\tau - (q_{i\mathcal{T}} + Q_{i\mathcal{T}-}) - w_\tau)\ q_{i\mathcal{T}}\right] \tag{B-19}$$

$$\text{subject to} \qquad w_\tau\,q_{i\mathcal{T}} \leq B, \qquad \text{for all } \omega \in \Omega. \tag{B-20}$$

Since the retailers are unable hedge the budget constraint in (B-20) must be imposed pathwise. As a result, problem (B-19)-(B-20) decouples and we can determine the retailers' optimal ordering strategy separately for each outcome $\omega \in \Omega$. Indeed, it is not hard to see that

$$q_{i\mathcal{T}} = \min\left\{\frac{(A_\tau - Q_{i\mathcal{T}-} - w_\mathcal{T})^+}{2} \;,\; \frac{B}{w_\mathcal{T}}\right\} \tag{B-21}$$

solves (B-19)-(B-20). In what follows, and without loss of optimality, we will assume that $w_\mathcal{T} \leq A_\tau$ for otherwise $q_i = 0$ for all $i = 1, \ldots, N$ and the supply chain would effectively shut down.

Let $Q_\mathcal{T} = \sum_{i=1}^N q_{i\mathcal{T}}$ be the cumulative order quantity in the retail market. Then since we will always have $A_\tau - Q_\mathcal{T} - w_\mathcal{T} \geq 0$ in equilibrium one can show that in equilibrium the optimality condition (B-21) is equivalent to

$$q_{i\mathcal{T}} = \min\left\{A_\tau - Q_\mathcal{T} - w_\mathcal{T} \;,\; \frac{B}{w_\mathcal{T}}\right\}. \tag{B-22}$$

Since (B-22) holds for all i we can sum it w.r.t. i and after simplifying obtain

$$Q_\mathcal{T}(w_\tau, B, N) = N \, \min\left\{\frac{A_\tau - w_\tau}{N+1}, \frac{B}{w_\tau}\right\} \tag{B-23}$$

as the Cournot solution for this benchmark.

The producer's problem is then to determine the optimal wholesale price menu, i.e. to solve $\Pi_{\mathrm{P}} = \max_{w_\tau} \mathbb{E}\Big[(w_\mathcal{T} - c_\tau)\, Q_\mathcal{T}(w_\tau, B, N)\Big]$. As with the retailers' problem, this optimization problem can be solved pathwise. In particular, we need to solve

$$\Pi_{\mathrm{P}|\mathcal{T}} = \max_{w_\mathcal{T}} \,(w_\tau - c_\mathcal{T})\, Q_\mathcal{T}(w_\tau, B, N). \tag{B-24}$$

The complete solution of the problem is presented in the following proposition.

Proposition 7 *Let $\delta_\mathcal{T} := \max\left\{c_\tau, \sqrt{(A_\tau^2 - 4\,(N+1)B)^+}\right\}$. The solution to the Cournot-Stackelberg game when retailers have no access to the financial markets satisfies:*

$$w_\tau = \frac{A_\tau + \delta_\mathcal{T}}{2}, \qquad q_\mathcal{T} = \frac{A_\tau - \delta_\mathcal{T}}{2\,(N+1)},$$

$$\Pi_{\mathrm{R}} = \mathbb{E}\left(\frac{A_\tau - \delta_\mathcal{T}}{2(N+1)}\right)^2 \qquad \textit{and} \quad \Pi_{\mathrm{P}} = \frac{N}{4\,(N+1)}\mathbb{E}[(A_\tau + \delta_\mathcal{T} - 2c_\tau)(A_\tau - \delta_\mathcal{T})].$$

The proof of this result is quite straightforward and is therefore omitted.

B.2 Decentralized Supply Chain with Hedging but No Borrowing

We now consider the benchmark in which the retailers are only able to use the financial markets for hedging. In particular, borrowing is not possible. We can view this as a special case of the general model presented in Section 2 in which the retailers' cost of debt is prohibitively high, i.e. $\underline{r} = \infty$. The next result follows directly[17] from Proposition 3. Though we don't state them in the corollary below, the optimal hedging gains also follow immediately from Proposition 3.

[17] Cases I to III of Proposition 3 are combined here in a single case. Note that the expression for Π_{P} in the statement of Corollary 2 is consistent with the corresponding expression in Case III of the proposition. This is seen by noting that in Case III we have the inequality defining $\phi^*(B)$ in (19) holding with equality. We can use this equality to substitute for B in the first term of the expression for Π_{P} in Case III. From this we immediately recover the expression for Π_{P} Corollary 2.

Corollary 2 *Suppose the retailers can only use the financial markets for hedging. Recalling the definition of $\phi^*(B)$ in (19), the Cournot-Stackelberg equilibrium of the game is given by*

$$w_\tau = \frac{A_\tau + \phi^*(B)\, c_\tau}{2} \quad \textit{and} \quad q_\tau = \frac{(A_\tau - \phi^*(B)\, c_\tau)^+}{2(N+1)}.$$

The players' expected payoffs satisfy

$$\Pi_{\mathrm{P}} \;=\; \frac{N}{4\,(N+1)}\,\mathbb{E}\big[(A_\tau + \phi^*(B)\, c_\tau - 2\, c_\tau)\,(A_\tau - \phi^*(B)\, c_\tau)^+\big] \quad \textit{and} \quad \Pi_{\mathrm{R}} = \frac{\mathbb{E}\big[((A_\tau - \phi^*(B)\, c_\tau)^+)^2\big]}{4(N+1)^2}.$$

We can also use the same approach to obtain the Cournot equilibrium in thus case. In particular, it is given by in Proposition 2 but with $\underline{r} = \infty$ there.

B.3 Decentralized Supply Chain with Borrowing but No Hedging

We now turn to the case in which the retailers can only use the financial market to borrow but are unable to engage in any hedging activity. In this setting, retailer i's best response is derived by solving

$$\begin{aligned} \Pi_{\mathrm{R}_i}(w_\tau, Q_{i\tau-}) \quad &= \quad \max_{q_{i\tau}, D_{i\tau}} \mathbb{E}[(A_\tau - (q_{i\tau} + Q_{i\tau-}) - w_\tau)\, q_{i\tau} - r_\tau\, D_{i\tau}] \\ \text{subject to} \quad & \quad w_\tau\, q_{i\tau} \le B + D_{i\tau}, \qquad \text{for all } \omega \in \Omega, \\ & \quad q_{i\tau} \ge 0 \quad \text{and} \quad D_{i\tau} \ge 0. \end{aligned}$$

In the absence of hedging, the budget constraint must be imposed pathwise and the optimization problem decouples for every $\omega \in \Omega$. Moreover for a given order quantity $q_{i\tau}$, the optimal level of debt is trivially $D^*_{i\tau} = (w_\tau\, q_{i\tau} - B)^+$. Substituting this back into the optimization problem above, we can remove the budget constraint and solve for the optimal order quantity $q^*_{i\tau}$. The solution is given by

$$q^*_{i\tau} = \begin{cases} \frac{(A_\tau - Q_{i\tau-} - (1+r_\tau)\, w_\tau)^+}{2} & \text{if } \frac{B}{w_\tau} \le \frac{(A_\tau - Q_{i\tau-} - (1+r_\tau)\, w_\tau)^+}{2} & \text{(Case 1)} \\ \frac{B}{w_\tau} & \text{if } \frac{(A_\tau - Q_{i\tau-} - (1+r_\tau)\, w_\tau)^+}{2} \le \frac{B}{w_\tau} \le \frac{(A_\tau - Q_{i\tau-} - w_\tau)^+}{2} & \text{(Case 2)} \\ \frac{(A_\tau - Q_{i\tau-} - w_\tau)^+}{2} & \text{if } \frac{(A_\tau - Q_{i\tau-} - w_\tau)^+}{2} \le \frac{B}{w_\tau}. & \text{(Case 3)} \end{cases} \tag{B-25}$$

We use this best-response strategy to characterize the symmetric Cournot equilibrium in the retailers' market for a given wholesale price menu w_τ.

Proposition 8 *Given a wholesale price w_τ, a symmetric Cournot equilibrium in the retailers' market is given by*

$$(q^*_\tau, D^*_\tau) = \begin{cases} \left(\frac{(A_\tau - (1+r_\tau)\, w_\tau)^+}{N+1},\ \frac{w_\tau\,(A_\tau - (1+r_\tau)\, w_\tau)^+}{N+1} - B\right) & \textit{if } B \le \frac{w_\tau\,(A_\tau - (1+r)\, w_\tau)^+}{N+1} & \text{(Case 1)} \\ \left(\frac{B}{w_\tau},\ 0\right) & \textit{if } \frac{w_\tau\,(A_\tau - (1+r_\tau)\, w_\tau)^+}{N+1} \le B \le \frac{w_\tau\,(A_\tau - w_\tau)^+}{N+1} & \text{(Case 2)} \\ \left(\frac{(A_\tau - w_\tau)^+}{N+1},\ 0\right) & \textit{if } \frac{w_\tau\,(A_\tau - w_\tau)^+}{N+1} \le B. & \text{(Case 3)} \end{cases}$$

Sketch Proof. Suppose, for example, Case 3 in (B-25) applies. It follows that $2q_{i\tau}^* = (A_\tau - Q_\tau + q_{i\tau}^* - w_\tau)^+$ from which it follows (after a little consideration of the role of the positive part) that $q_{i\tau}^* = (A_\tau - Q_\tau - w_\tau)^+$. Summing this last expression over i yields $Q_\tau = N(A_\tau - Q_\tau - w_\tau)^+$ from which it follows (after again considering the role of the positive part) $Q_\tau = N(A_\tau - w_\tau)^+/(N+1)$. Case 3 in the proposition then follows. Case 1 follows an almost identical argument while Case 2 requires a little more algebra to obtain. $\square$

Note that the solution for $q_{i\tau}^*$ and Proposition 8 above are almost identical to the results in Lemma 1 and Proposition 1. The only difference is that the budgets vary with i in Lemma 1 and Proposition 1.

Turning to the Stackelberg game, the producer selects the value of w by maximizing his expected profit. Specifically he must solve

$$\max_w \mathbb{E}[(w_\tau - c_\tau)\, Q_\tau^*], \quad \text{where} \quad Q_\tau^* := N\, q_\tau^*$$

where q_τ^* is given in Proposition 8. In solving this problem, we note the optimal w_τ^* can never be an interior or left-boundary point in Case 2 of Proposition 8. This follows because his objective in that region is monotonic increasing in $1/w_\tau$. As a result, we can search for w^* by restricting ourselves to an interior point in Case 1 or any solution in Case 3. We define

$$\overline{B}_\tau := \frac{A_\tau^2 - c_\tau^2}{4\,(N+1)} \qquad \text{and} \qquad \underline{B}_\tau = \frac{A_\tau^2 - \left(c_\tau + \sqrt{r_\tau\,(A_\tau^2 - (1+r_\tau)\, c_\tau^2)/(1+r_\tau)}\right)^2}{4\,(N+1)}.$$

The Stackelberg-Cournot equilibrium is then characterized by the following proposition which we state without proof.

Proposition 9 *A symmetric Stackelberg-Cournot equilibrium $(w_\tau^*, q_\tau^*, D_\tau^*)$ and corresponding firms' payoffs $(\Pi_{\mathrm{R}|\tau}^*, \Pi_{\mathrm{P}|\tau}^*)$ satisfy*

(a) Large Budget: *Suppose $B \geq \overline{B}_\tau$, then*

$$w_\tau = \frac{A_\tau + c_\tau}{2}, \qquad q_\tau = \frac{A_\tau - c_\tau}{2\,(N+1)}, \quad D_\tau^* = 0, \quad \Pi_{\mathrm{R}|\tau} = \frac{(A_\tau - c)^2}{4\,(N+1)^2} \quad \text{and} \quad \Pi_{\mathrm{P}|\tau} = \frac{N\,(A_\tau - c)^2}{4\,(N+1)}.$$

In this case, the retailers' budget constraints are not binding in equilibrium.

(b) Medium Budget: *Suppose $\underline{B}_\tau \leq B \leq \overline{B}_\tau$ then*

$$w_\tau = \frac{A_\tau + \sqrt{A_\tau^2 - 4\,(N+1)\,B}}{2}, \qquad q_\tau = \frac{A_\tau - \sqrt{A_\tau^2 - 4\,(N+1)\,B}}{2\,(N+1)}, \qquad D_\tau^* = 0,$$

$$\Pi_{\mathrm{R}|\tau} = \frac{\left(A_\tau - \sqrt{A_\tau^2 - 4\,(N+1)\,B}\right)^2}{4\,(N+1)^2} \quad \text{and} \quad \Pi_{\mathrm{P}|\tau} = N\,B - \frac{N\,c_\tau\left(A_\tau - \sqrt{A_\tau^2 - 4\,(N+1)\,B}\right)}{2\,(N+1)}.$$

In this case, the retailers's budget constraints are binding but no debt is used in equilibrium.

(c) Small Budget: *Suppose $B \leq \underline{B}_\tau$ then*

$$w_\tau = \frac{A_\tau + (1+r_\tau)\,c_\tau}{2\,(1+r_\tau)}, \qquad q_\tau = \frac{(A_\tau - (1+r_\tau)\,c_\tau)^+}{2\,(N+1)}, \qquad D_\tau = \frac{(A_\tau^2 - (1+r_\tau)^2\,c_\tau^2)^+}{4\,(N+1)\,(1+r_\tau)} - B,$$

$$\Pi_{\mathrm{R}|\tau} = \frac{((A_\tau - (1+r_\tau)\,c_\tau)^+)^2}{4\,(N+1)^2} + r_\tau\,B \qquad \text{and} \qquad \Pi_{\mathrm{P}|\tau}^* = \frac{N\,((A_\tau - (1+r_\tau)\,c_\tau)^+)^2}{4\,(N+1)(1+r_\tau)}.$$

In this case, the retailers' budget constraints are binding and debt is used in equilibrium.

B.4 Centralized Problem with Hedging and Borrowing

Finally[18], we consider the case in which the supply chain is controlled by a central planner (CP) that makes all of the decisions. As is customary in the supply chain management literature, we view this vertically integrated system as a benchmark to assess the inefficiencies of a decentralized system, in particular those arising from the double marginalization phenomenon induced by a two-tier system, i.e., retailers acting as middlemen, and the level of competition (or lack thereof) in the retailers' market. In order to have a fair comparison between our decentralized system and a vertically integrated one we will assume the CP is also budget constrained and endowed with a budget $B_{\mathrm{C}} := N \times B$.

We consider the case in which the CP has access to the financial markets and so she must solve:

$$\Pi_{\mathrm{C}} = \max_{Q_\tau \geq 0,\, G_\tau, D_\tau \geq 0} \mathbb{E}\left[(A_\tau - Q_\tau - c_\tau)\, Q_\tau - r_\tau\, D_\tau\right] \tag{B-26}$$

$$\text{subject to} \qquad c_\tau\, Q_\tau \leq B_{\mathrm{C}} + G_\tau + D_\tau, \qquad \text{for all } \omega \in \Omega \tag{B-27}$$

$$\mathbb{E}\left[G_\tau\right] = 0. \tag{B-28}$$

We make use of the following definitions in the statement of Proposition 10 below. We state it without proof as the result follows easily using Lagrange multiplier arguments.

$$\overline{Q}_{\mathrm{C}|\tau} := \frac{(A_\tau - c_\tau)^+}{2}, \qquad \overline{B}_{\mathrm{C}} := \mathbb{E}[c_\tau\, \overline{Q}_{\mathrm{C}|\tau}] \quad \text{and} \quad \underline{B}_{\mathrm{C}} = \mathbb{E}\left[c_\tau\, \frac{(A_\tau - c_\tau\,(1 + \underline{r}))^+}{2}\right].$$

Proposition 10 *The solution to the CP's problem (B-26)-(B-28) can be divided into three cases:*

- Case 1: *If $B_{\mathrm{C}} \geq \overline{B}_{\mathrm{C}}$ then the CP can select the optimal unconstrained production level $Q^*_\tau = \overline{Q}_{C|\tau}$ without using any debt, i.e. $D^*_\tau = 0$. Furthermore, there are infinitely many feasible hedging strategies that can implement this solution. One particular choice is*

$$G^*_\tau = (c_\tau\, \overline{Q}_{C|\tau} - B_{\mathrm{C}}) \left[1 + \left(\frac{\mathbb{E}[(c_\tau\, \overline{Q}_{C|\tau} - B_{\mathrm{C}})^+]}{\mathbb{E}[(B_{\mathrm{C}} - c_\tau\, \overline{Q}_{C|\tau})^+]} - 1\right) \mathbb{1}(B_{\mathrm{C}} \geq c_\tau\, \overline{Q}_{C|\tau})\right].$$

  The expected payoff of the CP is

$$\Pi_{\mathrm{C}} = \frac{\mathbb{E}[((A_\tau - c_\tau)^+)^2]}{4}.$$

- Case 2: *If $\underline{B}_{\mathrm{C}} \leq B_{\mathrm{C}} < \overline{B}_{\mathrm{C}}$ then the optimal production level is given by*

$$Q^*_\tau = \frac{(A_\tau - c_\tau\,(1 + \lambda))^+}{2}$$

  where $\lambda > 0$ solves the equation

$$B_{\mathrm{C}} = \mathbb{E}\left[c_\tau\, \frac{(A_\tau - c_\tau\,(1 + \lambda))^+}{2}\right].$$

[18] As mentioned at the beginning of this appendix, we didn't use the central planner formulation in the main text because of space constraints and so we only include it here for the sake of completion.

In this case, the CP does not raise any debt, i.e. $D_\tau^* = 0$, *and uses the hedging strategy*

$$G_\tau^* = c_\tau \, Q_\tau^* - B_{\mathrm{C}}.$$

The expected payoff of the CP is equal to

$$\Pi_{\mathrm{C}} = \frac{\mathbb{E}[((A_\tau^2 - c_\tau^2 \, (1+\lambda)^2)^+)]}{4} - B_{\mathrm{C}}.$$

- Case 3: *If* $B_{\mathrm{C}} < \underline{B}_{\mathrm{C}}$ *then the CP's optimal production level is*

$$Q_\tau^* = \frac{(A_\tau - c_\tau \, (1+\underline{r}))^+}{2}.$$

The CP only raises debt on $\mathcal{E}$ *where* $\mathcal{E} := \{r_\tau = \underline{r}\}$ *and there are infinitely many optimal borrowing strategies with the only requirement being that* $\mathbb{E}[D_\tau^*] = \mathbb{E}[c_\tau \, Q_\tau^*] - B_{\mathrm{C}}$. *One specific choice that borrows uniformly*[19] *in* $\mathcal{E}$ *is given by*

$$D_\tau^* = \left(\frac{\mathbb{E}[c_\tau \, Q_\tau^*] - B_{\mathrm{C}}}{\mathbb{P}(\mathcal{E})} \right) \mathbb{1}(r_\tau = \underline{r}).$$

Finally, the CP's optimal hedging strategy satisfies

$$G_\tau^* = c_\tau \, Q_\tau^* - B_{\mathrm{C}}$$

and her expected payoff is

$$\Pi_{\mathrm{C}} = \frac{\mathbb{E}[((A_\tau^2 - c_\tau^2 \, (1+\underline{r}_\tau)^2)^+)]}{4} - B_{\mathrm{C}}.$$

As Proposition 10 reveals, $\overline{Q}_{\mathrm{C}|\tau}$ is the CP's optimal production level if it had no budget constraint while $\overline{B}_{\mathrm{C}}$ corresponds to the minimum budget needed to implement this unconstrained production level. $\underline{B}_{\mathrm{C}}$ is the threshold budget level below which the CP's optimal solution requires borrowing.

C The Debt-Induced Discontinuity in the Cournot-Stackelberg Equilibrium

We now discuss in further detail the discontinuity in B that we observed in Figure 4. We will focus our discussion on the unconstrained equilibrium but we recall the static equilibrium in that figure also had a single point of discontinuity. We first emphasize that there is only one point of discontinuity and this occurs[20] at $\underline{B}_\tau$, the point at which the retailers switch from using costly debt to not using costly debt, i.e. the transition from region III to region IV in Proposition 3. The discontinuity in the retailers' expected profits at $\underline{B}_\tau$ occurs because w_τ^* is discontinuous at this point which in turns induces a discontinuity in q_τ^* and the retailers' expected profits.

[19] See footnote 7 on dividing by $\mathbb{P}(\mathcal{E})$ in the definition of D_τ^* when $\mathbb{P}(\mathcal{E}) = 0$.

[20] It can also be seen from the definition of $\overline{B}_\tau$ and ϕ^* that $\phi^*(\overline{B}_\tau) = 1$ so there is no discontinuity in moving from region II to region III in Proposition 3. That there is no discontinuity in moving from region I to region II in the proposition is immediate since the optimal quantities in those regions are independent of B.

Our first observation is to note that if the retailers were not allowed to access the debt market then the supply chain would shut down in the limit as $B \to 0$. This is clear from Figure 1. Indeed if access to the debt market is not possible and $B = 0$ then the budget constraint becomes $w_\tau\, q_{i\mathcal{T}} \leq G_{i\mathcal{T}}$ which must hold in every state. But since w_τ and $q_{i\mathcal{T}}$ are non-negative a.s., and $\mathbb{E}[G_{\mathcal{T}}] = 0$, it follows that $G_{\mathcal{T}} = 0$ a.s. and so $q_{i\mathcal{T}} = 0$ a.s., i.e. the supply chain shuts down.

Now note that the retailers do not use the debt markets in regions I to III of Proposition 3. In fact the retailers are effectively not budget constrained in regions I and II. In region III the retailers are budget constrained even after hedging. In this region the producer's expected profits satisfy

$$\begin{aligned} \Pi_{\mathrm{P}|\mathcal{T}} &= \mathbb{E}\left[(w^*_\tau - c_\tau)Q^*_{\mathcal{T}}\right] \\ &= NB - c_\tau \mathbb{E}\left[Q^*_{\mathcal{T}}\right] \end{aligned} \tag{C-29}$$

since the retailers' budget constraints are binding in this region and so $\mathbb{E}\left[w^*_\tau Q^*_{\mathcal{T}}\right] = NB$. It therefore follows from (C-29) that within this region (in which the retailers' budget constraints are binding) the producer would like to make Q^*_τ as small as possible in order to minimize his costs. He achieves this by choosing a w^*_τ in this region which results[21] in the optimal Lagrange multiplier for the retailers being zero.

As discussed above, however, the producer's expected profits will decrease to zero as B decreases to zero. This means there is some threshold level in which the producer will want to induce the retailers to borrow. This threshold is $\underline{B}_{\mathcal{T}}$ but as discussed immediately before Example **??**, the retailers' optimal Lagrange multiplier in the borrowing region, i.e. region IV where $B \leq \underline{B}_{\mathcal{T}}$, must be $\lambda^*(B) = \underline{r}_{\mathcal{T}}$. As a result there is a jump in the retailers' optimal Lagrange multiplier from 0 to $\underline{r}_{\mathcal{T}}$ at $\underline{B}_{\mathcal{T}}$. This in turn means the optimal price menu w^*_τ and total ordering quantity $Q^*_{\mathcal{T}}$ also jumps at this point.

Note also that within region IV the retailers' optimal ordering quantities no longer depend on B. This is clear from (14) when $\lambda^*(B) = \underline{r}_{\mathcal{T}}$. It therefore follows that the producer's optimal price menu w_τ is also independent of B in this region. The economic intuition for this is as follows. If the producer sets a price menu w_τ so that the retailers are willing to borrow when their budget is just below $\underline{B}_{\mathcal{T}}$, then this same price menu will also induce the retailers to borrow at all budgets $B < \underline{B}_{\mathcal{T}}$. This is because for a fixed price menu w_τ, the form of the price function implies that the marginal value to the retailers of an additional unit of budget is decreasing in B. Therefore if it's worthwhile borrowing at a budget of $\underline{B}_{\mathcal{T}}$ it will certainly also be worthwhile borrowing at all budgets $B < \underline{B}_{\mathcal{T}}$. Therefore in order to induce the retailers to borrow in this region the producer can set a price menu that induces the retailers to borrow at a budget just below $\underline{B}_{\mathcal{T}}$.

[21] In fact one can easily check that if we take w^*_τ as defined in Case 3 of Proposition 3 then it results in B being on the boundary of Cases 1 and 2 in Proposition 2 if $\phi^*(B) > 1$ and in the interior of Case 1 if $\phi^*(B) = 1$.