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To cite this article: Martin B. Haugh & Ashish Jain (2011) The dual approach to portfolio evaluation: a comparison of the static, myopic and generalized buy-and-hold strategies, Quantitative Finance, 11:1, 81-99, DOI: [10.1080/14697681003712870](https://doi.org/10.1080/14697681003712870)

To link to this article: <https://doi.org/10.1080/14697681003712870>



Published online: 01 Sep 2010.



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The dual approach to portfolio evaluation: a comparison of the static, myopic and generalized buy-and-hold strategies

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(Received 25 July 2007; in final form 18 February 2010)

We use the recently proposed duality approach to study the performance of *static*, *myopic* and *generalized buy-and-hold* (GBH) trading strategies. Our interest in static and GBH strategies is motivated by the fact that these strategies are intuitive and straightforward to implement in practice. The myopic strategy, while more difficult to implement, is often close to optimal and so we use it to obtain tight bounds on the performance of the true optimal dynamic trading strategy. We find that while this optimal dynamic strategy often significantly outperforms the GBH strategy, this is not true in general when no-borrowing or no-short-sales constraints are imposed on the investor. This has implications for investors when a dynamic trading strategy is too costly or difficult to implement in practice. For the class of security price dynamics under consideration, we also show that the optimal GBH strategy is always superior to the optimal static strategy. We also demonstrate that the dual approach is even more tractable than originally considered. In particular, we show it is often possible to solve for the theoretically satisfying upper bounds on the optimal value function that were suggested when the dual approach was originally proposed.

Keywords: Portfolio optimization; Duality; Generalized buy-and-hold strategy

1. Introduction

Ever since the pioneering work of Merton (1969, 1971), Samuelson (1969) and Hakansson (1970), considerable progress has been made in solving dynamic portfolio optimization problems. These problems are ubiquitous: individual agents, pension and mutual funds, insurance companies, endowments and other entities all face the fundamental problem of dynamically allocating their resources across different securities in order to achieve a particular goal. These problems are often very complex owing to their dynamic nature and high dimensionality, the complexity of real-world constraints, and parameter uncertainty. Using optimal control techniques, these researchers and others[†] solve for the optimal dynamic trading strategy under various price dynamics in frictionless markets.

Optimal control techniques dominated until martingale techniques were introduced by Karatzas *et al.* (1987) and Cox and Huang (1989). Under complete market assumptions, they showed how the portfolio choice problem could be decomposed into two subproblems. The first subproblem solved for the optimal terminal wealth, a problem which could be formulated as a static optimization problem given the complete markets assumption. The second subproblem then solved for the trading strategy that replicated the optimal terminal wealth. This new approach succeeded in expanding the class of dynamic problems that could be solved.

Dual methods were then used by a number of authors (Xu 1990, He and Pearson 1991a, b, Karatzas *et al.* 1991, Cvitanic and Karatzas 1992, Shreve and Xu 1992a, b) to extend the martingale approach to problems where markets are incomplete and agents face portfolio constraints.

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[†]See, for example, Merton (1990), Kim and Omberg (1996), and Liu (2008) and the many references cited therein.

Duality methods have since been very popular[†] for tackling other classes of portfolio optimization problems. These include, for example, problems with transaction costs and models where trading impacts security prices.

Applying some of these dual methods, Haugh, Kogan and Wang[‡] (2006) showed how suboptimal dynamic portfolio strategies could be evaluated by using them to compute lower and upper bounds on the expected utility of the true optimal dynamic trading strategy. In general, the better the suboptimal solution, the narrower the gap between the lower and upper bounds, and the more information you therefore have regarding how far the suboptimal strategy is from optimality. These techniques apply directly to multidimensional diffusion processes with incomplete markets and portfolio constraints such as no-short-selling or no-borrowing constraints.

The first goal of this paper is to use the dual-based portfolio evaluation technique of HKW to study the performance of the *static*, *myopic* and *generalized buy-and-hold* (GBH) trading strategies. A static strategy is a constant[§] proportion trading strategy where the agent invests the same constant proportion of his wealth in each of the risky assets at all times. A myopic strategy is a generalization of the static strategy whereby at each time t the agent solves the dynamic portfolio optimization problem assuming the instantaneous moments of assets returns will remain fixed at their current values for the remainder of the investment horizon. The static and myopic strategies are well known in the literature and further details and references will be provided in section 3.

We define the GBH strategies to be the class of strategies where the terminal wealth is a function of *only* the terminal security prices. In contrast, the terminal wealth of a static buy-and-hold strategy is always an *affine* function of terminal security prices. Haugh and Lo (2001) originally introduced these strategies in the context of static buy-and-hold portfolios that could invest at time $t = 0$ in a single risky stock, a cash account and European options[¶] on the stock. Because the payoffs of European options only depend on the terminal stock price, the expected utility of the optimal GBH strategy provided an upper bound on the expected utility of any static buy-and-hold strategy.

While the optimal GBH and static strategies will in general be inferior[⊥] to the optimal dynamic trading strategy, they are interesting in their own right as they are both intuitive and straightforward to implement in practice. For the price dynamics under consideration in this paper we will show that the optimal GBH strategy always outperforms the optimal static strategy. We use the myopic strategy and the dual approach of HKW to

compute tight bounds on the expected utility of the optimal dynamic trading strategy. We can then use these bounds to determine just how well the static and GBH strategies perform.

We find that when markets are incomplete and there are no portfolio constraints, the dynamic unconstrained optimal strategy often significantly outperforms the optimal GBH strategy. Once portfolio constraints are imposed, however, the GBH portfolio can often have a much higher expected utility than the optimal dynamic portfolio. In order to draw this conclusion it will be necessary to assume the existence of some non-constrained agents in the market-place who can ‘sell’ the GBH terminal wealth to constrained investors. While it is true that under this assumption there is nothing to stop these agents selling more general path-dependent portfolios to the constrained agents, we believe the simplicity of the GBH portfolios are more realistic and merit further study.

The second goal of this paper is to evaluate in further detail the dual-based approach of HKW and show that it is even more tractable than originally considered. We use a simple application of Ito’s Lemma to derive a closed-form solution for the optimal wealth and expected utility of this wealth when a static trading strategy is employed. Though this closed-form solution is particularly simple to derive, we have not seen it presented elsewhere and it has a number of applications. For example, it allows us to conclude that the optimal GBH strategy is always superior to the optimal static strategy. We can also use it to compute the precise and more theoretically satisfying upper bound that was originally proposed by HKW. Because the static strategy’s value function was unknown to HKW, they were forced to construct an alternative less satisfying upper bound.

The closed-form solution also allows us to define and compute an alternative optimal static trading strategy. This latter strategy corresponds to an agent who *knows* the true price dynamics but is restricted, for one reason or another, to employing a static trading strategy. This is in contrast to the usual definition of the optimal static strategy where the agent *chooses* a static strategy because he assumes that the investment opportunity set is not time-varying. Other applications of the closed-form solution that pertain to myopic strategies and minimizing dual-based upper bounds will be discussed in section 5. The remainder of this paper is organized as follows. Section 2 formulates the portfolio optimization problem while section 3 describes the three portfolio strategies. Section 4 reviews the portfolio duality theory and the dual approach for bounding the optimal expected utility

[†]See Rogers (2003a) for a survey of some of the more recent advances.

[‡]Hereafter referred to as HKW.

[§]One of the principal results of the early literature (Merton 1969, 1971, Samuelson 1969) is that a static trading strategy is optimal when the optimizing agent has constant relative risk aversion and security returns are independent and identically distributed.

[¶]Other researchers have also considered the problem of adding options to the portfolio optimization problem. See, for example, Evnine and Henriksson (1987) and Carr and Madan (2001).

[⊥]In related work, Kohn and Papazoglou (2004) identify those diffusion processes where the optimal dynamic trading strategy results in a terminal wealth that is a function of only the terminal security price. They do this in a complete markets setting.

for a given dynamic portfolio optimization problem. In section 5 we derive the terminal wealth and value function for any static strategy under our assumed price dynamics. Section 6 contains numerical results and we conclude in section 7. Most of the technical details are deferred to the appendices.

2. Problem formulation

In this section we formulate the dynamic portfolio optimization problem. We will follow the formulation of HKW.

2.1. The investment opportunity set and security price dynamics

We assume there are N risky assets and a single risk-free asset available in the economy. The time t vector of risky asset prices is denoted by $P_t = (P_t^{(1)}, \dots, P_t^{(N)})$ and the instantaneously risk-free rate of return is denoted by r , a constant.[†] Security price dynamics are driven by the M -dimensional vector of state variables X_t so that

$$\begin{aligned} dP_t &= P_t[\mu_P(X_t)dt + \Sigma_P dB_t], \\ dX_t &= \mu_X(X_t)dt + \Sigma_X dB_t, \end{aligned} \quad (1)$$

where $X_0 = 0$, $B_t = (B_{1t}, \dots, B_{Nt})$ is a vector of N independent Brownian motions, μ_P and μ_X are N - and M -dimensional drift vectors, and Σ_P and Σ_X are constant diffusion matrices of dimensions N by N and M by N , respectively. We assume that the diffusion matrix, Σ_P , of the asset return process is lower-triangular and non-degenerate so that $x^T \Sigma_P \Sigma_P^T x \geq \epsilon \|x\|^2$ for all x and some $\epsilon > 0$. Then we can define a process, η_t , as

$$\eta_t = \Sigma_P^{-1}(\mu_P(X_t) - r).$$

In a market without portfolio constraints, η_t corresponds to the market-price-of-risk process (e.g., Duffie (1996)). We make the standard assumption that the process η_t is square integrable so that[‡]

$$E_0 \left[\int_0^T \|\eta_t\|^2 dt \right] < \infty.$$

Note that return predictability[§] in the price processes in (1) is induced only through the drift vector, $\mu_P(X_t)$, and not the volatility, Σ_P . In this case it is well known[¶] that, in the absence of trading constraints, European option prices can be uniquely determined despite the fact that the market is incomplete. It is for precisely the same reason that any random variable, Z_T , can be attained as the terminal wealth of some dynamic trading strategy if Z_T is a function of only the terminal security prices. It is this fact that will allow us to use the straightforward

martingale technique to identify the optimal GBH strategy which we will define formally in section 3.3.

2.2. Portfolio constraints

A portfolio consists of positions in the N risky assets and the risk-free cash account. We denote the proportional holdings of the risky assets in the total portfolio value by $\theta_t = (\theta_{1t}, \dots, \theta_{Nt})$. The proportion in the risk-free asset is then given by $(1 - \theta_t^T \mathbf{1})$ where $\mathbf{1}$ is the unit vector of length N . To rule out arbitrage, we require the portfolio strategy to satisfy a square integrability condition, namely that $\int_0^T \|\theta_t\|^2 dt < \infty$ almost surely. The value of the portfolio, W_t , then has the following dynamics

$$\frac{dW_t}{W_t} = [(1 - \theta_t^T \mathbf{1})r + \theta_t^T \mu_P(X_t)]dt + \theta_t^T \Sigma_P dB_t. \quad (2)$$

We assume that the proportional holdings in the portfolio are restricted to lie in a closed convex set, K , that contains the zero vector. In particular, we assume that

$$\theta_t \in K. \quad (3)$$

If short sales are not allowed, for example, then the constraint set takes the form

$$K = \{\theta : \theta \geq 0\}. \quad (4)$$

If, in addition, borrowing is not allowed, then the constraint set takes the form

$$K = \{\theta : \theta \geq 0, \mathbf{1}^T \theta \leq 1\}. \quad (5)$$

2.3. Market incompleteness

In our setup, market incompleteness occurs for two reasons. (i) In the absence of portfolio constraints it occurs when the number of traded securities, N say, is smaller than the number, L say, of Brownian motions. This is the typical example of market incompleteness. (ii) Even when $N \geq L$, it can occur when portfolio constraints are imposed resulting in the inability to replicate certain payoffs. In fact, it is worth emphasizing that the first case can be viewed as a subset of the second case. In particular, when $N < L$ we can imagine adding $L - N$ assets to the economy so that the market becomes complete in the sense of (i). However, we can then insist that the holdings in these new $L - N$ assets must be zero, resulting in the incompleteness of (ii). We use this more general formulation in our model and hence take $L = N$ in (1).

In the numerical results of section 6, we will use the term ‘Incomplete markets’ to refer to case (i). Otherwise, we will refer to the ‘No Short Sales’ or ‘No Short Sales and No Borrowing’ cases even though the markets in these cases will be incomplete for both reasons (i) and (ii).

[†]The remark immediately following corollary 5.2 in section 5 outlines how we can easily handle stochastic interest rates.

[‡]Throughout the paper we will use $E_t[\cdot]$ to denote an expectation conditional on time t information.

[§]We say there is return predictability when the moments of the instantaneous returns depend on some observable state process, X_t .

[¶]Step 1 in appendix B explains why this is true.

2.4. Investor preferences

We assume that the portfolio policy is chosen to maximize the expected utility, $E_0[U(W_T)]$, of wealth at the terminal date T . The function $U(W)$ is assumed to be strictly monotone with positive slope, concave and smooth. It is assumed to satisfy the Inada conditions at zero and infinity so that $\lim_{W \rightarrow 0} U'(W) = \infty$ and $\lim_{W \rightarrow \infty} \frac{1}{W} U'(W) = 0$. In our numerical results, we assume the investor's preferences to be of the constant relative risk aversion (CRRA) type so that

$$U(W) = W^{1-\gamma}/(1-\gamma). \quad (6)$$

The investor's dynamic portfolio optimization problem is to solve for the value function, V_0 , at $t=0$ where

$$V_0 \equiv \sup_{\theta_t} E_0[U(W_T)], \quad (7)$$

subject to constraints (1), (2) and (3). We will not solve (7) but instead focus on the three strategies of section 3. This is because these strategies can be calculated in many high-dimensional problems when it is not possible to obtain the true optimal solution to (7).

3. Trading strategies

In this section we define three strategies that, in general, are suboptimal solutions to (7). They are: (i) the *static* trading strategy, (ii) the *myopic* trading strategy, and (iii) the *generalized buy-and-hold* or *GBH* strategy. We will use the static and GBH strategies to demonstrate how most of the work that is required for calculating upper bounds on the optimal value function can often be done analytically. We will also study the performance of these strategies as they are of interest in their own right. The static and GBH strategies are of particular interest as they are intuitive and easy to implement in practice. Though the myopic strategy is not so easy to implement in practice, it is often very close to optimal. As a result it can often serve as a proxy for the optimal strategy when the latter is not available to us. The dual approach is useful in implementing this program as it enables us to bound how far from optimality these strategies are.

3.1. The static trading strategy

A static strategy is a *constant proportion* trading strategy where the portfolio weights in the risky assets do not vary with time or changes in state variables. Despite its name, however, it is a dynamic strategy as the portfolio needs to be continuously re-balanced. The optimal static strategy

ignores the predictability of stock returns and is defined using the unconditional average returns, μ_0 , instead of the time-varying conditional expected returns on the stocks. The optimal static strategy may then be found as the solution to

$$\theta^{\text{static}} = \arg \max_{\theta \in K} (\mu_0^T - r)\theta - \frac{1}{2} \gamma \theta^T \Sigma_P \Sigma_P^T \theta. \quad (8)$$

In particular, under the optimal static trading strategy the agent re-balances his portfolio at each time t so that he always maintains a constant (vector) proportion, θ^{static} , of his time wealth invested in the risky assets. It is well known[†] that this static strategy is an optimal policy in a dynamic model with a constant investment opportunity set and cone[§] constraints on portfolio positions. In section 5 we will prove a simple proposition showing that, under the price dynamics in (1), the terminal wealth resulting from any static trading strategy is a function of only the terminal security prices.

Remark 1: We should emphasize that the static strategy is only optimal from the perspective of an agent who believes that the instantaneous average return, μ_t , is constant for all t and equal to μ_0 . In particular, an agent who knows the true price dynamics could choose a superior static strategy. See proposition 5.1 in section 5 for further details.

3.2. The myopic trading strategy

The *myopic* strategy is defined similarly to the static policy except that at every time, t , the instantaneous moments of asset returns are assumed to be fixed at their current values for the remainder of the investment horizon. In particular, at each time t the agent invests a (vector) proportion, θ_t^{myopic} , of his time t wealth in the risky assets where θ_t^{myopic} solves

$$\theta_t^{\text{myopic}} = \arg \max_{\theta \in K} (\mu_P^T(X_t) - r)\theta - \frac{1}{2} \gamma \theta^T \Sigma_P \Sigma_P^T \theta. \quad (9)$$

The myopic policy in (9) ignores the hedging component[¶] of the optimal trading strategy. In particular, at each time t the agent observes the instantaneous moments of asset returns, $\mu_P^T(X_t)$ and Σ_P , and, assuming that these moments are fixed from time t onwards, he solves for the optimal static trading strategy. As was the case with the static strategy, the optimization problem in (9) is obtained immediately from the HJB equation that the myopic investor formulates at each time t . Because we do not have a closed-form expression for the terminal wealth resulting from the myopic strategy, we can estimate its

[†]Though we will report the optimal solution in the case of no-trading constraints.

[‡]See Merton (1969,1971) or, more recently, section 6.6 of Karatzas and Shreve (1998).

[§]A portfolio is cone constrained if the portfolio weights must lie in a non-empty closed convex cone.

[¶]In fact, the hedging component of the optimal trading strategy at any point in time is often defined to be the difference between the optimal strategy and the myopic strategy. The myopic strategy therefore ignores the hedging component *by definition*. The reasoning behind this definition is that the myopic strategy captures that component of the optimal strategy that is explained by the current investment opportunity set. Any deviation from that component is due to inter-temporal hedging of changes in this investment opportunity set. See Merton (1990).

expected utility by simulating the stochastic differential equations for X_t , P_t and W_t , and solving (9) at each point on each simulated path.

The myopic strategy is popular[†] in the financial literature for several reasons. For example, it is expected that the myopic policy will be very close to optimal when the hedging component of the optimal trading strategy is not significant. This will often be true of the model parameters we use for the numerical experiments of section 6. The myopic strategy can also be used to estimate the magnitude of this hedging component. Moreover it is clear from (8) and (9) that it is as easy to solve for the myopic policy as it is for the static policy.

3.3. The generalized buy and hold (GBH) trading strategy

As stated earlier, a GBH strategy is any strategy resulting in a terminal wealth that is a function of only the terminal security prices. When the optimizing agent does not face any trading constraints, the optimal GBH strategy may be found by solving

$$V_0^{\text{gbh}} = \sup_{W_T} E_0[U(W_T)],$$

subject to the $\sigma(P_T)$ -measurability of W_T , (10)

where $\sigma(P_T)$ is defined to be the σ -algebra generated by the vector of terminal security prices, P_T . Moreover, under the price dynamics in (1), standard martingale methods may be used to identify the trading strategy that replicates the solution to (10), W_T^{gbh} .

When the agent does face trading constraints, the optimal GBH strategy is in general no longer attainable from a dynamic trading strategy. In this case we rely on the implicit assumption[‡] that there are other unconstrained agents in the marketplace who can replicate and therefore uniquely price any GBH strategy. The constrained agent is then assumed to ‘purchase’ his optimal GBH terminal wealth from one of these unconstrained agents. We then compare the expected utility of the agent’s optimal GBH terminal wealth to the optimal wealth that could be attained from a constrained dynamic trading strategy. We will see in section 6 that the GBH strategy often significantly outperforms the optimal constrained dynamic trading strategy.

One possible criticism of this analysis is to ask why the constrained agent should restrict himself to purchasing a GBH terminal wealth from an unconstrained agent. Instead, acting as though he was unconstrained, the agent could compute his optimal terminal wealth and then ‘purchase’ this wealth from one of the unconstrained

agents who can actually replicate it. While this criticism has some merit, we believe that the GBH strategies are simple to understand and, as a straightforward generalization of the well-known buy-and-hold strategy, deserve attention in their own right. Moreover, we believe that many investors care more about the final level of security prices, rather than the *path* of security prices, when they are evaluating their investment performance. They are aware that they generally do not possess market-timing skills but, at the same time, they do not wish to ‘miss the boat’ on a sustained bull market, for example. Clearly, access to generalized buy-and-hold strategies would be of particular interest to such investors.

3.4. Solving for the optimal GBH trading strategy

We now outline the steps required for computing and assessing the performance of the optimal GBH strategy. Further details are provided in appendix B where we specialize to the price dynamics assumed in section 6.

- (1) **Solving the SDE:** We first solve the stochastic differential equation (SDE) for the price processes, P_t , and state variable, X_t , under both the real-world probability measure, P , and any risk-neutral probability measure, Q . Recall that since markets are incomplete, a unique risk-neutral measure does not exist.
- (2) **Compute the conditional state price density:** We compute the state price density, π_T^b , conditional on the terminal security prices. In particular, we solve for

$$\pi_T^b = E_0[\pi_T | P_T^{(i)} = b_i, i = 1, \dots, N]. \quad (11)$$
 It is worth mentioning that while there are infinitely many state price-density processes, π_t , we can use any such process on the right-hand side of (11) and obtain the same§ conditional state price density, π_T^b . Note that π_T^b can be interpreted as the date $t=0$ price of having \$1 at time T in any state for which $P_T = b$.
- (3) **Compute optimal GBH wealth:** We then use the static martingale approach to solve for the optimal GBH strategy. In particular, we solve

$$V_0^{\text{gbh}} = \sup_{W_T} E_0 \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right], \quad \text{subject to } E_0[\pi_T^b W_T] = W_0. \quad (12)$$

Note that because of our use of the conditional state price density, π_T^b , in (12), we do not need to explicitly impose the constraint that W_T be $\sigma(P_T)$ -measurable. This constraint will be satisfied automatically.

[†]See, for example, Kroner and Sultan (1993), Lioui and Poncet (2000), Brooks *et al.* (2002) and Basak and Chabakauri (2008)

[‡]A similar assumption is often used to justify complete-market models and the Black–Scholes model, in particular, for pricing options. For example, a common criticism of these models states that if markets are complete then we shouldn’t need derivative securities in the first place. The response to this is that the market is complete only for a small subset of agents whose presence allows us to uniquely price derivatives. For the majority of investors, the presence of trading frictions and constraints implies that derivative securities do add to the investment opportunity set.

[§]This is consistent with our earlier observation that all European options prices can be uniquely determined despite the market incompleteness.

- (4) **Determine the value function at all intermediate times:** Compute the GBH value function, V_t^{gbh} , for all $t \in [0, T]$. This is effectively the same problem that we solve in step 3.
- (5) **Determine the replicating trading strategy:** Once the optimal GBH wealth, W_T^{gbh} , has been determined, we compute the replicating strategy, θ_t^{gbh} , that attains W_T^{gbh} .
- (6) **Compute lower and upper bounds on the global optimal value function:** Using the GBH trading strategy, θ_t^{gbh} , and the GBH value function, V_t^{gbh} , we can compute an upper bound on the optimal value function, V_0 , for the problem in (7). This step is done using the duality-based algorithm described in section 4. Note that V_0^{gbh} constitutes a lower bound on the optimal value function. If the lower and upper bounds are close to one another, then we can conclude that the optimal GBH strategy is indeed close to the true optimal solution. When the GBH investor faces trading constraints we will compare V_0^{gbh} to the optimal value function, V_0 , that results from dynamic trading with constraints.

4. Review of duality theory and construction of upper bounds

In this section we review[†] the duality approach of HKW for analysing the quality of a suboptimal strategy. This is done by using the suboptimal strategy to construct a lower and upper bound on the true value function. If the difference between the two bounds is large, i.e. the duality gap is wide, then it suggests that the suboptimal policy is not close to the optimal solution. If the duality gap is narrow, then (i) we know that the suboptimal strategy is close to optimal and (ii) we know approximately the optimal value function.

Starting with the portfolio optimization problem of section 2, we can define a fictitious problem $(\mathcal{P}^{(v)})$, based on a different financial market and without the portfolio constraints. First we define the *support function* of \mathbf{K} , $\delta(\cdot)$: $\mathbb{R}^N \rightarrow \mathbb{R} \cup \infty$, by setting

$$\delta(v) = \sup_{x \in \mathbf{K}} (-v^\top x). \quad (13)$$

The effective domain of the support function is denoted by $\tilde{\mathbf{K}} := \{v \in \mathbf{K} : \delta(v) < \infty\}$. Because the constraint set \mathbf{K} is convex and contains zero, the support function is continuous and bounded from below on its effective domain $\tilde{\mathbf{K}}$. Letting $\{\mathcal{F}_t\}$ denote the standard Brownian filtration, we then define the set \mathbf{D} of (\mathcal{F}_t) -adapted \mathbb{R}^N -valued processes to be

$$\mathbf{D} = \left\{ v_t, 0 \leq t \leq T : v_t \in \tilde{\mathbf{K}}, E_0 \left[\int_0^T \delta(v_t) dt \right] + E_0 \left[\int_0^T \|v_t\|^2 dt \right] < \infty \right\}. \quad (14)$$

For each process v in \mathbf{D} , we define a fictitious market $M^{(v)}$. In this market, one can trade the N stocks and the risk-free cash account. The diffusion matrix of stock returns in $M^{(v)}$ is the same as in the original market. However, the risk-free rate and the vector of expected stock returns are different. In particular, the risk-free rate process and the market price of risk in the fictitious market are defined respectively by

$$r_t^{(v)} = r + \delta(v_t), \quad (15a)$$

$$\eta_t^{(v)} = \eta_t + \Sigma_P^{-1} v_t, \quad (15b)$$

where $\delta(v)$ is the support function defined in (13). We assume that $\eta_t^{(v)}$ is square-integrable. Following Cox and Huang (1989), the state-price density process $\pi_t^{(v)}$ in the fictitious market is given by

$$\pi_t^{(v)} = \exp \left(- \int_0^t r_s^{(v)} ds - \frac{1}{2} \int_0^t \eta_s^{(v)\top} \eta_s^{(v)} ds - \int_0^t \eta_s^{(v)\top} dB_s \right), \quad (16)$$

and the vector of expected returns is given by

$$\mu_{P_t}^{(v)} = r_t^{(v)} + \Sigma_P \eta_t^{(v)}.$$

The dynamic portfolio choice problem in the fictitious market without position constraints can be equivalently formulated in a static form:[‡]

$$V_0^{(v)} \equiv \sup_{\{W_T\}} E_0[U(W_T)], \text{ subject to } E_0[\pi_T^{(v)} W_T] \leq W_0. \quad (\mathcal{P}^{(v)})$$

Due to its static nature, the problem $(\mathcal{P}^{(v)})$ is easy to solve. For example, when the utility function is of the CRRA type with relative risk aversion γ so that $U(W) = W^{1-\gamma}/(1-\gamma)$, the corresponding value function in the fictitious market is given explicitly by

$$V_0^{(v)} = \frac{W_0^{1-\gamma}}{1-\gamma} E_0[\pi_T^{(v)(\gamma-1)/\gamma}]^\gamma. \quad (17)$$

It is easy to see that, for any admissible choice of $v \in \mathbf{D}$, the value function in (17) gives an upper bound for the optimal value function of the original problem. In the fictitious market, the wealth dynamics of the portfolio are given by

$$dW_t^{(v)} = W_t^{(v)} [(r_t^{(v)} + \theta_t^\top \Sigma_P \eta_t^{(v)}) dt + \theta_t^\top \Sigma_P dB_t], \quad (18)$$

so that

$$\begin{aligned} \frac{dW_t^{(v)}}{W_t^{(v)}} - \frac{dW_t}{W_t} &= [(r_t^{(v)} - r_t) + \theta_t^\top \Sigma_P (\eta_t^{(v)} - \eta_t)] dt \\ &= (\delta(v_t) + \theta_t^\top v_t) dt. \end{aligned}$$

The last expression is non-negative according to (13) since $\theta_t \in \mathbf{K}$. Therefore, $W_t^{(v)} \geq W_t \forall t \in [0, T]$ and so

$$V_0^{(v)} \geq V_0. \quad (19)$$

[†]We borrow from HKW in much of our review.

[‡]See Karatzas *et al.* (1987), Cox and Huang (1989), or section 6 of Karatzas and Shreve (1998).

Under fairly general assumptions, it can be shown that there exists a process, v^* , such that (19) holds with equality. While one can pick any fictitious market from the admissible set \mathbf{D} to compute an upper bound, HKW showed how a given suboptimal strategy, $\tilde{\theta}_t$, may be used to select a particular $\hat{v}_t \in \mathbf{D}$. If the suboptimal strategy is in fact optimal, then the lower bound associated with the suboptimal strategy will equal the associated upper bound, thereby demonstrating its optimality. We will now describe how $\hat{v}_t \in \mathbf{D}$ can be chosen.

In order to construct the fictitious market as defined by \hat{v}_t , we first use the optimal solution to the dual problem to establish the link between the optimal policy θ^* and value function $V^{(v^*)}$, and the corresponding fictitious asset price processes, as defined by v^* . According to the dynamic programming principle, at any time t

$$V_t = V_t^{(v^*)} = \sup_{\{W_T\}} E_t[U(W_T)], \quad \text{subject to } E_t\left[\frac{\pi_T^{(v^*)}}{\pi_t^{(v^*)}} W_T\right] \leq W_t.$$

The first-order optimality conditions then imply $U'(W_T) = \lambda_t \pi_T^{(v^*)} / \pi_t^{(v^*)}$ where λ_t is the Lagrange multiplier on the time t budget constraint. On the other hand, the envelope condition implies[†] that the partial derivative of the value function with respect to the portfolio value satisfies $\partial V_t / \partial W_t = \lambda_t$. We therefore see that $\pi_T^{(v^*)} / \pi_t^{(v^*)} = U'(W_T) / (\partial V_t / \partial W_t)$ and so we obtain

$$\frac{\pi_s^{(v^*)}}{\pi_t^{(v^*)}} = \frac{\partial V_s / \partial W_s}{\partial V_t / \partial W_t}, \quad \forall s \geq t.$$

The above equality implies that

$$d \ln \pi_t^{(v^*)} = d \ln \frac{\partial V_t}{\partial W_t}, \quad (20)$$

so that, in particular, the stochastic part of $d \ln \pi_t^{(v^*)}$ equals the stochastic part of $d \ln \partial V_t / \partial W_t$. If V_t is smooth, Itô's lemma and equations (16) and (2) imply that

$$\eta_t^{(v^*)} = -W_t \frac{\partial^2 V_t / \partial W_t^2}{\partial V_t / \partial W_t} \Sigma_P^\top \theta_t^* - \left(\frac{\partial V_t}{\partial W_t} \right)^{-1} \Sigma_{X_t}^\top \left(\frac{\partial^2 V_t}{\partial W_t \partial X_t} \right), \quad (21)$$

where θ_t^* denotes the optimal portfolio policy for the original problem.

Given an approximation to the optimal portfolio policy, $\tilde{\theta}_t$, one can compute the corresponding approximation to the value function, \tilde{V}_t , defined as the conditional expectation of the utility of terminal wealth, under the portfolio policy $\tilde{\theta}_t$. If analytic expressions are not available for $\tilde{\theta}_t$ and \tilde{V}_t , then approximations to the portfolio policy and value function can be obtained using a variety of methods (e.g., Brandt *et al.* 2005 and Haugh and Jain 2007). In this paper, we will calculate $\tilde{\theta}_t$ for the static, myopic and GBH strategies, and use each one in turn to construct an upper bound on the unknown true value function, V_0 . Assuming that the approximate value function \tilde{V}_t is sufficiently smooth, we can replace

V_t and θ_t^* in (21) with \tilde{V}_t and $\tilde{\theta}_t$ and obtain

$$\tilde{\eta}_t = -W_t \left(\frac{\partial_{WW} \tilde{V}_t}{\partial W \tilde{V}_t} \right) \Sigma_P^\top \tilde{\theta}_t - (\partial_W \tilde{V}_t)^{-1} \Sigma_{X_t}^\top (\partial_{WX} \tilde{V}_t), \quad (22)$$

where ∂_W denotes the partial derivative with respect to W , and ∂_{WX} and ∂_{WW} are corresponding second partial derivatives. We then define \tilde{v}_t as a solution to (15b) where $\eta_t^{(v)}$ is replaced by $\tilde{\eta}_t$.

In the special but important case of a CRRA utility function the expression for $\tilde{\eta}_t$ simplifies. In the case of a CRRA utility function, for a given trading strategy, $\tilde{\theta}_t$, the corresponding value function is of the form

$$\tilde{V}_t = g(t, X_t) \frac{W_t^{1-\gamma}}{1-\gamma}.$$

Hence, the candidate market-price-of-risk in the dual problem simplifies to

$$\begin{aligned} \tilde{\eta}_t &= \gamma \Sigma_P^\top \tilde{\theta}_t - \frac{\Sigma_X^\top}{\tilde{V}_t} \left(\frac{\partial \tilde{V}_t}{\partial X_t} \right) \\ &= \gamma \Sigma_P^\top \tilde{\theta}_t - \frac{\Sigma_X^\top}{g(t, X_t)} \left(\frac{\partial g(t, X_t)}{\partial X_t} \right), \end{aligned} \quad (23)$$

where γ is the relative risk aversion coefficient of the utility function. One therefore only needs to compute the first derivative of the value function with respect to the state variables, X_t , to evaluate the second term in (23). This simplifies the numerical implementation if the value function and its derivatives need to be calculated numerically, since it is easier to estimate first-order partial derivatives than second-order partial derivatives. In the case of the static trading strategy, the analytic expression of proposition 5.1 will enable us to compute an analytic expression for the partial derivatives. For the GBH trading strategy we can also compute the value function and its derivatives analytically. These calculations are given in appendix B. But for more general strategies such as the myopic strategy and others, we generally do not have an analytical solution for the value function and its derivatives available to us.

Obviously, $\tilde{\eta}_t$ is a candidate for the market price of risk in the fictitious market. However, there is no guarantee that $\tilde{\eta}_t$ and the corresponding process \tilde{v}_t belong to the feasible set \mathbf{D} defined by (14). In fact, for many important classes of problems the support function $\delta(v_t)$ may be infinite for some values of its argument. We therefore look for a price-of-risk process $\hat{\eta}_t \in \mathbf{D}$ that is 'close' to $\tilde{\eta}_t$ by formulating a simple quadratic optimization problem with linear constraints. Depending on the portfolio constraints, this problem may be solved analytically. Otherwise, we solve it numerically at each discretization point on each simulated path of the underlying SDEs. The lower bound is then computed by simulating the given portfolio strategy. The same simulated paths of the SDEs are then used to estimate the upper bound given by (17). At each discretization point on each simulated

[†]See Karatzas and Shreve (1998), section 3.7, theorem 7.7 for a formal proof.

path we solve a quadratic optimization problem to find the appropriate $\hat{\eta}_t \in \mathbf{D}$. See HKW for further details.

It is worth mentioning that when HKW were computing the upper bounds corresponding to the static and myopic strategies, they only used the first term on the right-hand side of (23) as an expression for the second term was unavailable. While the resulting bounds were still valid upper bounds, they were not the precise bounds as prescribed by their algorithm. In this paper, proposition 5.1 in section 5 allows us to use both terms[†] on the right-hand side of (23) to derive the upper bound corresponding to the static strategy. We can also calculate both terms on the right-hand side of (23) for the GBH strategy.

5. An explicit solution for the static strategy

We have the following proposition which solves for the terminal wealth corresponding to any static trading strategy.

Proposition 5.1: *Suppose price dynamics satisfy (1) and a static trading strategy is followed so that at each time $t \in [0, T]$ a proportion, θ , of time t wealth is invested in the risky assets with $1 - \theta^\top \mathbf{1}$ invested in the risk-free asset. Then the terminal wealth, W_T , resulting from this strategy only depends on the terminal prices of the risky assets, P_T . In particular, we have*

$$W_T = W_0 \exp \left((1 - \theta^\top \mathbf{1})rT + \frac{1}{2} \theta^\top (\text{diag}(\Sigma_P \Sigma_P^\top) - \Sigma_P \Sigma_P^\top) T + \theta^\top \left[\ln \frac{P_T}{P_0} \right] \right). \quad (24)$$

Proof: See appendix A. □

While straightforward to derive and perhaps not particularly surprising, we have not seen the statement of proposition 5.1 elsewhere. Moreover, it has a number of applications. First, the static strategy is typically used as a base case when researchers study the value of predictability in security prices. Since predictability is often induced via the drift term as in (1), the expression in (24) applies. This means that the expected utility of the static strategy can often be determined in closed form when the distribution of P_T is also known. For example, if $\log(P_t)$ is a (vector) Gaussian process, then W_T is log-normally distributed and

$$V_t^{\text{static}} = E_t[W_T^{1-\gamma}/(1-\gamma)] \quad (25)$$

can be computed analytically. This is obviously much more efficient than computing V_t^{static} numerically by simulating the underlying stochastic differential equations for X_t , P_t and W_t .

This latter simulation approach was used by HKW when using the static strategy to compute lower and upper bounds on the expected utility, V_0 , of the true optimal dynamic trading strategy. Moreover, because the analytic expression for V_t^{static} in (24) was unavailable, HKW were unable to compute the more theoretically satisfying upper bound on V_0 that uses both terms on the right-hand side of (23). Using (24), it is straightforward to compute these two terms and then use Monte Carlo to determine the corresponding upper bound. This is the second application of proposition 5.1.

Third, the ability to compute V_t^{static} analytically also implies that the *true* optimal static strategy, $\theta^{\text{static_opt}}$, can be found by directly maximizing V_t^{static} over θ instead of solving (8). This would be the strategy chosen by an investor who *knows* the true price dynamics in the market but who is restricted, for one reason or another, to choosing a static strategy. In section 6 we will briefly compare the performance of $\theta^{\text{static_opt}}$ with the static strategy, θ^{static} , defined in (8). We will see that $\theta^{\text{static_opt}}$ can often significantly outperform[‡] θ^{static} . That said, we will use θ^{static} as our ‘optimal’ static strategy with which we will compare the GBH and myopic strategies. We do this for two reasons: (i) in order to be consistent with HKW and the usual definition of the optimal static strategy and (ii) because we will not have the analytic solution of proposition 5.1 available to us for more general price processes. In such circumstances we would have to use θ^{static} anyway. It is also worth mentioning that proposition 5.1 would allow us to define an alternative version of the myopic strategy where at each time t we maximize (25) assuming the instantaneous moments of asset returns are held fixed from time t onwards. Again for reasons of consistency, however, the numerical results of section 6 were derived using the myopic policy as defined in section 3.2.

Finally, because the terminal wealth of the optimal static strategy depends only on the terminal security prices, we also have the following corollary.

Corollary 5.2: *Assuming the price dynamics in (1), a utility-maximizing agent will always prefer the optimal GBH strategy to any static strategy.*

Remark 2: We have assumed a constant risk-free rate in (24) but we could easily relax this assumption by assuming, for example, a Gaussian process for r_t . If $\log(P_t)$ was also Gaussian we would again obtain that

[†]As in HKW, we continue to omit the second term in (23) when computing the upper bound corresponding to the myopic strategy. Haugh and Jain (2007), however, show how cross-path regressions and path-wise estimators can be used to efficiently estimate this second term. Moreover, their numerical results show that the duality gap for the myopic strategy can be reduced by approximately 50% when this second term is included. When we ignore the second term (as we do in this paper for the myopic strategy) we can still often conclude that the myopic strategy is very close to optimal. This follows when the lower and upper bounds are very close to each other.

[‡]In some cases the improvement in the continuously compounded certainty equivalent return can be as large as 1.2% per annum. In other cases, the improvement is negligible.

Table 1. Calibrated model parameters. The four sets of model parameters correspond to: (1) size sorted portfolios and the dividend yield as a state variable; (2) size sorted portfolios and the term spread as a state variable; (3) book-to-market sorted portfolios and the dividend yield as a state variable; (4) book-to-market sorted portfolios and the term spread as a state variable. Parameter values are based on the estimates in tables 1 and 2 of Lynch (2001).

k	μ_0	μ_1	Σ_P			
<i>Parameter set 1</i>						
0.366	0.081	0.034	0.186	0.000	0.000	0.000
	0.110	0.059	0.228	0.083	0.000	0.000
	0.130	0.073	0.251	0.139	0.069	0.000
	0.000	0.000	−0.741	−0.037	−0.060	0.284
<i>Parameter set 2</i>						
1.671	0.081	0.046	0.186	0.000	0.000	0.000
	0.110	0.070	0.227	0.082	0.000	0.000
	0.130	0.086	0.251	0.139	0.069	0.000
	0.000	0.000	−0.017	0.149	0.058	1.725
<i>Parameter set 3</i>						
0.366	0.142	0.065	0.256	0.000	0.000	0.000
	0.109	0.049	0.217	0.054	0.000	0.000
	0.089	0.049	0.207	0.062	0.062	0.000
	0.000	0.000	−0.741	0.040	0.034	0.288
<i>Parameter set 4</i>						
1.671	0.142	0.061	0.256	0.000	0.000	0.000
	0.109	0.060	0.217	0.054	0.000	0.000
	0.089	0.067	0.206	0.062	0.062	0.000
	0.000	0.000	−0.017	0.212	0.096	1.716

W_T was log-normally distributed and the value function could again be computed analytically. The GBH strategy and value function could also be calculated analytically but corollary 5.2 would no longer hold.

5.1. Minimizing the upper bound

It is also possible to solve for the static strategy that minimizes the upper bound. This can be done by using (23) and, once again, proposition 5.1. This is of interest because (i) it helps demonstrate the general tractability of the dual approach in many circumstances and (ii) it is of interest to see whether strategies that optimize (over some class of suboptimal strategies) the lower and upper bounds coincide. In appendix C we do this for the particular set of dynamics assumed in section 6. We find in the case of incomplete markets that there are infinitely many static strategies that minimize the upper bound, none of which coincide with the static strategy of (8).

6. Numerical results

We use the same[†] model specification as that of HKW who in turn specify their model as a continuous time version of the market model of Lynch (2001). In particular, our model dynamics are as specified in (1), but now we assume that there are three risky securities and one state variable so that $N=4$ and $M=1$.

We assume the drift of the asset returns, $\mu(X_t)$, is an affine function of the single state variable, X_t , which follows a mean reverting Ornstein–Uhlenbeck process. Hence, it is an incomplete market model even in the absence of portfolio constraints. The asset return dynamics satisfy

$$\begin{aligned} r_t &= r, \\ dP_t &= P_t[(\mu_0 + X_t\mu_1)dt + \Sigma_P dB_t^P], \\ dX_t &= -kX_t dt + \Sigma_X dB_t^P. \end{aligned} \quad (26)$$

The second equation specifies the dynamics of the three traded risky securities. The diffusion matrix Σ_X is of size 1 by 4 and coincides with the last row of matrix Σ_P . The vectors μ_0 and μ_1 define the drift vector for the risky securities. The third equation specifies the dynamics of the state variable, X_t , whose initial value is set to zero in all of the numerical examples. Recall that in section 2 we assumed the number of risky securities, N , was equal to the number of Brownian motions. We maintain consistency with this by simply regarding X_t as the price process of an additional risky security in which we are always constrained to keep a zero[‡] position.

Lynch considered two choices for the state variable: (i) the dividend yield and (ii) the term spread. The dividend yield captures the rate at which dividends are paid out as a fraction of the total stock market value. The term spread is the difference in yields between 20 year and one month Treasury securities. Both of these predictive variables are normalized to have zero mean and unit variance. Lynch also considered two sets of risky

[†]By using the same model specification, we can also compare the performance of the static upper bound that we compute using both terms from the right-hand side of (23) with the upper bound computed in HKW that only used the first term.

[‡]This is a standard way of embedding the incomplete markets problem into the more general portfolio constraints problem. See the discussion of market incompleteness in section 2.

Table 2. Incomplete markets. This table reports the results for parameter sets 1 and 2. The parameter sets are defined in table 1. The rows marked LB^s , LB^m and LB^{gbh} report estimates of the expected utility achieved by using the static portfolio strategy, myopic portfolio strategy and generalized buy-and-hold (GBH) portfolio strategy, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked UB^s , UB^m and UB^{gbh} report the estimates of the upper bound on the true value function computed from the static, myopic and generalized buy-and-hold (GBH) portfolio strategies, respectively. The row marked V^u reports the optimal value function for the problem.

	$T = 5$			$T = 10$		
	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$
<i>Parameter set 1</i>						
LB^s	7.49 (7.47, 7.51)	4.63 (4.62, 4.64)	3.27 (3.27, 3.28)	7.70 (7.68, 7.70)	4.84 (4.83, 4.85)	3.42 (3.42, 3.43)
UB^s	9.44 (9.40, 9.49)	5.93 (5.86, 6.00)	4.15 (4.02, 4.27)	10.12 (10.09, 10.15)	6.61 (6.51, 6.71)	4.64 (4.38, 4.91)
LB^m	9.35 (9.32, 9.37)	5.72 (5.70, 5.73)	3.97 (3.96, 3.98)	9.94 (9.92, 9.96)	6.18 (6.16, 6.19)	4.29 (4.28, 4.29)
UB^m	9.46 (9.42, 9.50)	5.96 (5.89, 6.04)	4.18 (4.06, 4.30)	10.14 (10.11, 10.17)	6.67 (6.57, 6.78)	4.71 (4.44, 4.99)
LB^{gbh}	8.36 (8.31, 8.41)	5.29 (5.24, 5.35)	3.75 (3.70, 3.80)	8.68 (8.65, 8.72)	5.83 (5.78, 5.87)	4.22 (4.17, 4.27)
UB^{gbh}	9.42 (9.35, 9.49)	5.91 (5.80, 6.02)	4.16 (4.01, 4.31)	10.13 (10.08, 10.18)	6.64 (6.53, 6.75)	4.78 (4.59, 4.98)
V^u	9.44	5.95	4.19	10.09	6.62	4.75
<i>Parameter set 2</i>						
LB^s	6.54 (6.51, 6.57)	3.68 (3.67, 3.70)	2.59 (2.58, 2.60)	6.52 (6.50, 6.54)	3.67 (3.66, 3.68)	2.58 (2.57, 2.58)
UB^s	9.14 (9.10, 9.19)	5.00 (4.95, 5.05)	3.38 (3.33, 3.43)	9.33 (9.29, 9.36)	5.06 (5.04, 5.12)	3.43 (3.38, 3.48)
LB^m	9.02 (8.99, 9.05)	4.92 (4.90, 4.93)	3.33 (3.32, 3.34)	9.10 (9.08, 9.12)	4.95 (4.94, 4.97)	3.35 (3.34, 3.36)
UB^m	9.20 (9.15, 9.24)	5.09 (5.04, 5.14)	3.47 (3.42, 3.53)	9.38 (9.35, 9.41)	5.18 (5.14, 5.22)	3.52 (3.47, 3.57)
LB^{gbh}	6.55 (6.51, 6.59)	3.70 (3.68, 3.72)	2.60 (2.59, 2.62)	6.53 (6.50, 6.56)	3.69 (3.68, 3.71)	2.60 (2.59, 2.61)
UB^{gbh}	9.19 (9.14, 9.24)	5.02 (4.94, 5.10)	3.38 (3.30, 3.47)	9.58 (9.52, 9.63)	5.20 (5.13, 5.27)	3.50 (3.42, 3.58)
V^u	9.02	4.92	3.33	9.08	4.94	3.34

assets: (i) portfolios obtained by sorting stocks on their size and (ii) portfolios obtained by sorting stocks on their book-to-market ratio. The two choices of risky assets and the two choices of the predictive variable result in four sets of calibrated parameter values. These are reported in table 1. We set the risk-free rate, r , equal to 0.01 throughout.

As mentioned earlier, we assume that the utility function is of the constant relative risk aversion (CRRA) type so that $U(W) = W^{(1-\gamma)}/(1-\gamma)$. We consider three values for the relative risk aversion parameter $\gamma = 1.5, 3$, and 5 . We consider two values for the time horizon: $T = 5$ and $T = 10$ years. When simulating the SDEs we use 100 discretization points per year.

We consider three types of market constraints: (i) the base case where the agent does not face any trading constraints, (ii) the agent faces no-short-sales and no-borrowing constraints, and (iii) the agent faces no-short-sales constraints. In the first case we evaluate the static, myopic and GBH strategies by computing their value functions. While the value functions for the static and GBH strategies can be computed analytically, the value

function of the myopic strategy as well as all of the upper bounds had to be computed by numerically by simulating the SDEs. For reasons of consistency, we therefore estimated all of the bounds by simulating the SDEs. Since these trading strategies are feasible dynamic trading strategies, their value functions constitute valid lower bounds on the value function of the optimal dynamic trading strategy. We also report the value of this optimal value function in the base case as it can be computed explicitly using the results of Kim and Omberg (1996).

Though the optimal value function is available, we also use the three suboptimal strategies to compute upper bounds on this optimal value function. There are two reasons for doing this. First, ours is the first study that can compute the exact[†] upper bounds prescribed by the algorithm of HKW and it would be interesting to see how they vary with the quality of the lower bounds. Moreover, in the case of the static strategy, it is of interest to see how the upper bound here compares with the upper bound reported in HKW. Second, when dynamic constraints are imposed the optimal value function is no longer available and so it is necessary to compute upper bounds in

[†]This is only the case for the static and GBH strategies, but see Haugh and Jain (2007) for the myopic strategy.

Table 3. Incomplete markets. This table reports the results for parameter sets 3 and 4. The parameter sets are defined in table 1. The rows marked LB^s , LB^m and LB^{gbh} report estimates of the expected utility achieved by using the static portfolio strategy, myopic portfolio strategy and generalized buy-and-hold (GBH) portfolio strategy, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked UB^s , UB^m and UB^{gbh} report the estimates of the upper bound on the true value function computed from the static, myopic and generalized buy-and-hold (GBH) portfolio strategies, respectively. The row marked V^u reports the optimal value function for the problem.

	$T = 5$			$T = 10$		
	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$
<i>Parameter set 3</i>						
LB^s	14.65 (14.62, 14.68)	8.58 (8.56, 8.60)	5.73 (5.72, 5.74)	15.03 (15.01, 15.05)	8.98 (8.96, 8.99)	6.01 (6.01, 6.02)
UB^s	16.79 (16.73, 16.85)	10.27 (10.15, 10.39)	6.96 (6.76, 7.17)	17.78 (17.74, 17.82)	11.52 (11.36, 11.68)	8.09 (7.68, 8.50)
LB^m	16.64 (16.61, 16.68)	9.88 (9.86, 9.90)	6.61 (6.59, 6.62)	17.47 (17.44, 17.49)	10.60 (10.58, 10.61)	7.10 (7.09, 7.12)
UB^m	16.81 (16.76, 16.87)	10.34 (10.22, 10.46)	7.04 (6.83, 7.25)	17.82 (17.78, 17.87)	11.67 (11.51, 11.83)	8.26 (7.84, 8.69)
LB^{gbh}	15.74 (15.65, 15.82)	9.64 (9.53, 9.76)	6.57 (6.46, 6.70)	16.23 (16.13, 16.34)	10.75 (10.63, 10.88)	7.63 (7.51, 7.76)
UB^{gbh}	16.78 (16.68, 16.87)	10.36 (10.19, 10.53)	7.18 (6.92, 7.44)	17.80 (17.73, 17.87)	11.65 (11.46, 11.86)	8.39 (8.01, 8.78)
V^u	16.79	10.32	7.06	17.76	11.55	8.12
<i>Parameter set 4</i>						
LB^s	13.16 (13.12, 13.19)	7.09 (7.07, 7.11)	4.65 (4.64, 4.67)	13.14 (13.12, 13.17)	7.08 (7.06, 7.10)	4.65 (4.63, 4.66)
UB^s	16.06 (16.00, 16.13)	8.54 (8.46, 8.62)	5.53 (5.44, 5.62)	16.28 (16.24, 16.33)	8.66 (8.58, 8.73)	5.60 (5.50, 5.71)
LB^m	15.91 (15.87, 15.95)	8.45 (8.42, 8.47)	5.48 (5.46, 5.49)	16.02 (15.99, 16.05)	8.51 (8.49, 8.54)	5.51 (5.49, 5.53)
UB^m	16.08 (16.01, 16.14)	8.57 (8.49, 8.65)	5.56 (5.47, 5.65)	16.30 (16.25, 16.35)	8.69 (8.61, 8.76)	5.64 (5.53, 5.74)
LB^{gbh}	13.17 (13.11, 13.23)	7.11 (7.07, 7.14)	4.66 (4.64, 4.69)	13.13 (13.09, 13.18)	7.10 (7.06, 7.13)	4.66 (4.63, 4.68)
UB^{gbh}	16.14 (16.04, 16.25)	8.55 (8.42, 8.68)	5.51 (5.36, 5.66)	16.55 (16.47, 16.63)	8.74 (8.61, 8.87)	5.60 (5.42, 5.78)
V^u	15.91	8.45	5.47	16.00	8.49	5.49

order to determine how far the suboptimal strategies are from optimality. Since we therefore need to report upper bounds when trading constraints are imposed, for the sake of consistency we will also report upper bounds even when there are no trading constraints. For each of the three suboptimal strategies, their associated upper bounds are computed by simulating the underlying SDEs. At each discretization point on each simulated path, we solve a simple quadratic optimization[†] problem in order to solve for the market-price-of-risk process in the associated fictional market. See section 4 and HKW for further details.

When trading constraints are imposed, we again use the static and myopic strategies to compute lower and upper bounds on the true optimal value function. There is nothing new here[‡] over and beyond what is already presented in HKW. However, the principal goal of this section is to compare the optimal GBH strategy with the static and myopic strategies. Recall that when trading constraints are imposed we assume that the agent can

purchase the optimal GBH wealth from an unconstrained agent in the market place. We therefore display§ the results for the optimal GBH strategy alongside the lower and upper bounds for the constrained static and myopic strategies.

In all of our results, we report the expected utility as the continuously compounded certainty equivalent return, R . The value of R corresponding to a value function, V_0 , is defined by $U(W_0 e^{RT}) = V_0$. Finally, we mention again that the static strategy we consider is θ^{static} as defined by equation (8). In section 6.4 we will briefly compare the performance of θ^{static} with the true optimal static strategy, $\theta^{\text{static_opt}}$, as defined in section 5.

6.1. Incomplete markets

We first consider the case of incomplete markets where there are no trading constraints. Tables 2 and 3 report the estimates of the expected utility under the static, myopic and GBH portfolio strategies as well as their

[†]Only the subproblem corresponding to no short sales needs to be solved numerically. The subproblems corresponding to (i) no constraints and (ii) no-borrowing and no-short-sales constraints can both be solved analytically.

[‡]Except for how we computed the upper bound associated with the static strategy as mentioned earlier.

[§]While the GBH strategy could be used to compute a valid upper bound for the value function of the optimal constrained dynamic strategy, we do not bother to do so as the upper bound would not correspond to any feasible suboptimal strategy.

Table 4. No short sales and no borrowing. The four parameter sets are defined in table 1. The rows marked LB^s , LB^m and V^{gbh} report estimates of the expected utility achieved by using the static, myopic and GBH portfolio strategies, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked UB^s and UB^m report the estimates of the upper bound on the true value function.

	$T=5$			$T=10$		
	$\gamma=1.5$	$\gamma=3$	$\gamma=5$	$\gamma=1.5$	$\gamma=3$	$\gamma=5$
<i>Parameter set 1</i>						
LB^s	7.38 (7.36, 7.40)	4.63 (4.62, 4.64)	3.27 (3.27, 3.28)	7.56 (7.55, 7.57)	4.84 (4.83, 4.85)	3.42 (3.42, 3.43)
UB^s	8.93 (8.91, 8.96)	7.36 (7.34, 7.39)	6.88 (6.85, 6.90)	9.21 (9.19, 9.23)	7.94 (7.93, 7.96)	7.62 (7.60, 7.63)
LB^m	8.09 (8.07, 8.11)	5.57 (5.56, 5.59)	3.91 (3.89, 3.92)	8.35 (8.34, 8.36)	5.99 (5.98, 6.00)	4.22 (4.21, 4.23)
UB^m	8.15 (8.12, 8.18)	5.84 (5.79, 5.90)	4.18 (4.08, 4.28)	8.45 (8.43, 8.47)	6.48 (6.42, 6.54)	4.77 (4.64, 4.94)
V^{gbh}	8.36 (8.31, 8.41)	5.29 (5.24, 5.35)	3.75 (3.70, 3.80)	8.68 (8.65, 8.72)	5.83 (5.78, 5.87)	4.22 (4.17, 4.27)
<i>Parameter set 2</i>						
LB^s	6.53 (6.51, 6.56)	3.68 (3.67, 3.70)	2.59 (2.58, 2.60)	6.51 (6.49, 6.53)	3.67 (3.66, 3.68)	2.58 (2.57, 2.58)
UB^s	9.56 (9.53, 9.60)	6.84 (6.80, 6.88)	5.57 (5.52, 5.61)	9.69 (9.66, 9.72)	6.95 (6.91, 6.98)	5.66 (5.62, 5.70)
LB^m	7.88 (7.86, 7.90)	4.85 (4.84, 4.87)	3.30 (3.29, 3.31)	7.91 (7.90, 7.93)	4.87 (4.86, 4.88)	3.31 (3.30, 3.32)
UB^m	7.93 (7.89, 7.96)	4.98 (4.93, 5.03)	3.42 (3.37, 3.48)	7.95 (7.93, 7.98)	5.01 (4.97, 5.04)	3.44 (3.39, 3.49)
V^{gbh}	6.55 (6.51, 6.59)	3.70 (3.68, 3.72)	2.60 (2.59, 2.62)	6.53 (6.50, 6.56)	3.69 (3.68, 3.71)	2.60 (2.59, 2.61)
<i>Parameter set 3</i>						
LB^s	10.02 (10.00, 10.04)	6.75 (6.74, 6.76)	4.61 (4.60, 4.62)	10.21 (10.20, 10.22)	7.11 (7.10, 7.12)	4.87 (4.86, 4.87)
UB^s	10.51 (10.48, 10.54)	8.71 (8.67, 8.74)	7.91 (7.87, 7.94)	10.75 (10.73, 10.77)	9.32 (9.30, 9.34)	8.78 (8.76, 8.80)
LB^m	10.17 (10.15, 10.18)	7.65 (7.63, 7.66)	5.35 (5.33, 5.36)	10.37 (10.36, 10.38)	8.18 (8.17, 8.19)	5.80 (5.79, 5.81)
UB^m	10.23 (10.20, 10.26)	8.05 (7.98, 8.11)	5.85 (5.69, 6.01)	10.47 (10.46, 10.49)	8.88 (8.82, 8.94)	6.79 (6.34, 7.25)
V^{gbh}	15.74 (15.65, 15.82)	9.64 (9.53, 9.76)	6.57 (6.46, 6.70)	16.23 (16.13, 16.34)	10.75 (10.63, 10.88)	7.63 (7.51, 7.76)
<i>Parameter set 4</i>						
LB^s	9.21 (9.18, 9.23)	5.28 (5.27, 5.30)	3.55 (3.54, 3.56)	9.19 (9.18, 9.21)	5.27 (5.25, 5.28)	3.54 (3.53, 3.55)
UB^s	10.23 (10.19, 10.26)	7.60 (7.54, 7.65)	5.74 (5.68, 5.80)	10.30 (10.27, 10.32)	7.67 (7.63, 7.72)	5.80 (5.74, 5.85)
LB^m	9.51 (9.49, 9.53)	6.07 (6.05, 6.09)	4.05 (4.04, 4.06)	9.51 (9.50, 9.53)	6.08 (6.06, 6.09)	4.06 (4.05, 4.07)
UB^m	9.54 (9.51, 9.57)	6.20 (6.15, 6.26)	4.19 (4.12, 4.25)	9.55 (9.52, 9.57)	6.22 (6.17, 6.26)	4.20 (4.14, 4.26)
V^{gbh}	13.17 (13.11, 13.23)	7.11 (7.07, 7.14)	4.66 (4.64, 4.69)	13.13 (13.09, 13.18)	7.10 (7.06, 7.13)	4.66 (4.63, 4.68)

corresponding upper bounds on the optimal dynamic trading strategy. As demonstrated by HKW, the myopic strategy outperforms the static strategy in that the former has a higher lower bound. Perhaps surprisingly, however, we see that the upper bound generated by the static strategy is now superior, i.e. lower, than the upper bound generated by the myopic strategy. This occurs because the static upper bound is generated using both terms on the right-hand side of (23), whereas the myopic upper bound used only the first term. In fact, HKW showed that in the case of incomplete markets, the static and myopic upper bounds will coincide when they are both generated using only the first term of (23), so it is not surprising that the static upper bound is superior when calculated using

both terms. Haugh and Jain (2007), however, show that the myopic upper bound does marginally better than the static bound when the second term is estimated and included using Monte-Carlo techniques.

Confirming the results of corollary 5.2, we see that the GBH strategy always outperforms the optimal static strategy, significantly so in the cases of parameter sets 1 and 3. It is no surprise that the optimal myopic strategy generally outperforms the optimal GBH strategy as the former strategy can take explicit advantage of the variability in the state variable, X_t . However, it is quite surprising that in the case of the third parameter set, we see that the GBH strategy outperforms the myopic strategy when $T=10$ and $\gamma=3$ or $\gamma=5$. The upper bound

Table 5. No short sales. The four parameter sets are defined in table 1 and the problem horizon is $T = 5$ years. The rows marked LB^s , LB^m and V^{gbh} report the estimates of the expected utility achieved by using the static, myopic and GBH portfolio strategies, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked UB^s and UB^m report the estimates of the upper bound on the true value function.

	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$
<i>Parameter set 1</i>			
LB^s	7.46 (7.43, 7.49)	4.62 (4.60, 4.63)	3.26 (3.25, 3.27)
UB^s	9.98 (9.91, 10.05)	6.01 (5.90, 6.13)	4.16 (4.00, 4.33)
LB^m	9.04 (8.96, 9.12)	5.58 (5.53, 5.62)	3.89 (3.86, 3.92)
UB^m	9.14 (9.02, 9.27)	5.77 (5.56, 5.98)	4.04 (3.75, 4.33)
V^{gbh}	8.36 (8.31, 8.41)	5.29 (5.24, 5.35)	3.75 (3.70, 3.80)
<i>Parameter set 2</i>			
LB^s	6.52 (6.47, 6.56)	3.68 (3.66, 3.70)	2.59 (2.57, 2.60)
UB^s	10.65 (10.56, 10.73)	5.75 (5.65, 5.85)	3.72 (3.62, 3.82)
LB^m	8.92 (8.82, 9.01)	4.86 (4.81, 4.91)	3.30 (3.26, 3.33)
UB^m	9.01 (8.87, 9.15)	4.98 (4.82, 5.14)	3.38 (3.20, 3.56)
V^{gbh}	6.55 (6.51, 6.59)	3.70 (3.68, 3.72)	2.60 (2.59, 2.62)
<i>Parameter set 3</i>			
LB^s	11.14 (11.11, 11.18)	6.74 (6.72, 6.76)	4.61 (4.59, 4.62)
UB^s	14.06 (13.97, 14.14)	8.48 (8.33, 8.64)	5.85 (5.62, 6.09)
LB^m	12.80 (12.71, 12.89)	7.82 (7.77, 7.88)	5.34 (5.30, 5.38)
UB^m	12.99 (12.83, 13.14)	8.22 (7.95, 8.49)	5.73 (5.35, 6.11)
V^{gbh}	15.74 (15.65, 15.82)	9.64 (9.53, 9.76)	6.57 (6.46, 6.70)
<i>Parameter set 4</i>			
LB^s	9.69 (9.64, 9.74)	5.29 (5.26, 5.32)	3.56 (3.54, 3.58)
UB^s	13.55 (13.46, 13.65)	7.26 (7.14, 7.37)	4.71 (4.59, 4.84)
LB^m	11.36 (11.25, 11.47)	6.10 (6.04, 6.16)	4.04 (4.00, 4.08)
UB^m	11.47 (11.30, 11.63)	6.21 (6.02, 6.41)	4.09 (3.86, 4.33)
V^{gbh}	13.17 (13.11, 13.23)	7.11 (7.07, 7.14)	4.66 (4.64, 4.69)

computed from the GBH strategy is generally comparable to the upper bound generated from the myopic strategy despite the fact that the myopic strategy is usually significantly better than the GBH strategy. This is probably a reflection of our failure to include the second term on the right-hand side of (23) when estimating the myopic upper bound.

6.2. No-short-sales and no-borrowing constraints

Table 4 reports the results for when short sales and borrowing are prohibited. We see that the performance of the static and myopic strategies often deteriorates considerably. This is particularly true for parameter sets 3 and 4 and, not surprisingly, is more pronounced for lower values of risk aversion. Of particular interest is how the GBH strategy performs in relation to the myopic strategy. We see in the cases of parameter sets 1 and 2 that the myopic strategy still outperforms the GBH strategy, though not by a significant amount. In the case of parameter sets 3 and 4, however, the GBH portfolio significantly outperforms the myopic strategy. The out-performance is on the order of four or five percentage points per annum in the case of $\gamma = 1.5$, and one to two percentage points otherwise. This is significant and it is clear that constrained investors would easily prefer to purchase the optimal GBH strategy rather than implementing a dynamic constrained strategy. Note that in table 4 we do not display an upper bound generated by the GBH strategy. While it is straightforward to construct such an upper bound, we emphasize again that there does not exist a self-financing trading strategy that generates the GBH terminal wealth when trading constraints are imposed.

6.3. No-short-sales constraints

Table 5 reports the results when only a no-short-sales constraint is imposed. For this problem, the quadratic optimization problem that we must solve at each discretization point on each simulated path needs to be solved numerically. This is in contrast[†] to the earlier two cases where the quadratic optimization problem had an analytic solution. Due to the increased computational burden when solving for the upper bound, we only consider $T = 5$ in this case.

We draw the same conclusions as we did for the case where no-short-sales and no-borrowing constraints were imposed. The myopic strategy still outperforms the GBH strategy using parameter sets 1 and 2 but the GBH strategy outperforms under parameter sets 3 and 4. The extent of the GBH strategy's out-performance (one or two percentage points per annum) is not as great since the myopic strategy is now less constrained. However, investors who are free to borrow but are still constrained by the inability to short-sell would still clearly prefer to purchase the GBH portfolio when parameter sets 3 or 4 prevail.

6.4. Comparing the 'optimal' static strategies

While not the focus of this paper, it is interesting to note the difference in performance between θ^{static} and $\theta^{\text{static_opt}}$, as defined in sections 3.2 and 5, respectively. Recall that θ^{static} is the optimal strategy of an investor who assumes that the moments of instantaneous returns are constant through time and equal to their long-term average. Such an investor will select this strategy as his optimal strategy.

[†]We have not discussed the specific details of these quadratic optimization problems in this paper. HKW describes these problems in some detail and how the precise problem depends on the portfolio constraints that are imposed.

Table 6. Comparison of optimal static strategies in incomplete markets. The four parameter sets are defined in table 1. The rows marked $LB(\theta^{\text{static}})$ report the expected utility achieved by the static strategy as defined by equation (8). The rows marked $LB(\theta^{\text{static_opt}})$ report the expected utility achieved by the true optimal static strategy as defined in section 5. Expected utility is reported as a continuously compounded certainty equivalent return.

	$T = 5$			$T = 10$		
	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 5$
<i>Parameter set 1</i>						
$LB(\theta^{\text{static}})$	7.49	4.63	3.27	7.69	4.84	3.42
$LB(\theta^{\text{static_opt}})$	7.61	4.93	3.55	7.90	5.36	3.93
<i>Parameter set 2</i>						
$LB(\theta^{\text{static}})$	6.54	3.69	2.59	6.52	3.67	2.58
$LB(\theta^{\text{static_opt}})$	6.55	3.70	2.60	6.53	3.68	2.59
<i>Parameter set 3</i>						
$LB(\theta^{\text{static}})$	14.66	8.58	5.73	15.06	8.97	6.01
$LB(\theta^{\text{static_opt}})$	14.94	9.26	6.37	15.51	10.19	7.21
<i>Parameter set 4</i>						
$LB(\theta^{\text{static}})$	13.16	7.09	4.66	13.16	7.09	4.66
$LB(\theta^{\text{static_opt}})$	13.16	7.09	4.66	13.16	7.09	4.66

On the other hand, $\theta^{\text{static_opt}}$ is chosen by the investor who knows the true price dynamics but is forced to select a static trading strategy. It is clear that $\mathbf{E}_0[U(\theta^{\text{static_opt}})] \geq \mathbf{E}_0[U(\theta^{\text{static}})]$. The extent to which $\theta^{\text{static_opt}}$ outperforms θ^{static} can be seen in table 6 where we consider only† the incomplete market case. It is interesting to note that in the cases of parameter sets 2 and 4 there is practically no difference between the two strategies. This is not true of parameter sets 1 and 3, however, where the difference can range from 12 basis points per annum to 1.2% per annum.

7. Conclusions and further research

For a particular class of security price dynamics, we obtained a closed-form solution for the terminal wealth and expected utility of the classic *constant proportion* or static trading strategy. We then used this solution to study in further detail the portfolio evaluation approach recently proposed by Haugh, Kogan and Wang (2006) (HKW). In particular, we solved for the more theoretically satisfying upper bound on the optimal value function that was originally proposed by HKW. We also used this result to show that the upper bound could be minimized analytically over the class of static trading strategies.

For the same class of security price dynamics, we solved for the optimal GBH strategy and showed that in some circumstances it is comparable, in terms of expected utility, to the optimal dynamic trading strategy. Moreover, when the optimizing agent faces dynamic trading constraints such as no-short-sales or no-borrowing constraints, the optimal GBH strategy can often significantly outperform the optimal constrained dynamic trading strategy. This has implications for investors when: (i) a dynamic trading strategy is too costly or difficult to implement in practice and (ii) when the optimal GBH portfolio can be purchased from an unconstrained agent. We also concluded that the

optimal GBH strategy is superior to θ^{static} in that it achieves a higher expected utility.

There are several possible directions for future research. First, it would be interesting to extend the analysis to other security price dynamics. Are there other price processes, for example, where moderately risk averse investors with long time horizons might prefer the GBH strategy to the optimal myopic strategy? We saw this to be the case with parameter set 3, even when dynamic trading constraints were not imposed.

Another direction for future research is to continue the study of duality methods for assessing suboptimal portfolio strategies that are easy to implement in practice. For example, instead of trying to solve for a dynamic trading strategy we might instead define a set of parameterized strategies with a view to optimizing over this set. This would involve a static optimization problem and therefore be much easier to solve. Dual methods would then play the role of determining whether the optimal parameterized strategy is far from the global optimum. A related application of the duality method is in determining the value of information. For example, as a generalization of the GBH strategies we could allow any wealth that is a function of the entire paths of the traded securities *only*. Such a wealth, $W_T^{\text{gbh-path}}$ say, could not depend explicitly on the paths of the predictive processes, X_t . The difference between the expected utility of the global optimal strategy and the optimal $W_T^{\text{gbh-path}}$ could then be interpreted as representing the value of observing the evolution of X_t . This problem is also related to filtering problems where X_t is an unobserved latent variable. Clearly the dual method could be useful in investigating these issues.

Finally, it would be interesting to develop primal-dual style algorithms for finding good suboptimal policies. The work of Haugh *et al.* (2006) is a basic attempt in this direction. They use approximate dynamic

†We could also have reported results for the cases where portfolio constraints are imposed. The out-performance would be less noticeable in these cases.

programming (ADP) methods to construct trading strategies that are then evaluated using the dual-based portfolio evaluation approach. Their algorithm does not constitute a primal-dual algorithm, however, in that the dual formulation is not used to construct the trading strategy. It is worth noting that Brandt *et al.* (2005) were the first to use ADP methods to develop good dynamic trading strategies but their algorithm does not extend easily to problems with portfolio constraints. More recently, Brown *et al.* (2007) generalized the dual approach of Rogers (2003a) and Haugh and Kogan (2004) for solving optimal stopping problems. Their dual approach is based on information relaxations and it can be used to find approximate solutions and bounds to general finite-horizon discrete-time dynamic programs. We believe these methods could be a particularly profitable direction for future research into dynamic portfolio optimization problems.

Acknowledgements

We thank two anonymous referees for comments that greatly improved the exposition of this paper.

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Appendix A: The static strategy and proof of proposition 5.1

Proof (proof of proposition 5.1): Using (1) and applying Itô’s lemma to $\ln P_T$ we obtain

$$\begin{aligned} \ln P_T = \ln P_0 + \int_0^T \left(\mu_P(X_t) - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^\top) \right) dt \\ + \int_0^T \Sigma_P dB_t^P. \end{aligned} \quad (\text{A1})$$

The wealth dynamics for a static trading strategy, θ , are given by (2) with θ_t replaced by θ . A simple application of Itô's lemma to $\ln W_T$ then implies

$$W_T = W_0 \exp \left(\int_0^T ((1 - \theta^\top \mathbf{1})r + \theta^\top \mu_P(X_t)) dt - \frac{1}{2} (\theta^\top \Sigma_P \Sigma_P^\top \theta) T + \theta^\top \Sigma_P B_T^P \right). \quad (\text{A2})$$

Substituting (A1) into (A2) we then obtain

$$W_T = W_0 \exp \left((1 - \theta^\top \mathbf{1})rT + \frac{1}{2} \theta^\top (\text{diag}(\Sigma_P \Sigma_P^\top) - \Sigma_P \Sigma_P^\top \theta) T + \theta^\top \left[\ln \frac{P_T}{P_0} \right] \right), \quad (\text{A3})$$

as desired. \square

Under the price dynamics assumed in section 6, it is easy to see that under the physical probability measure, P , the terminal security prices, P_T , are multivariate log-normally distributed. In particular, $Y := \ln(P_T) \sim N(\mu_Y, \Sigma_Y)$ where μ_Y and Σ_Y are given by equations (B6) and (B7), respectively. It therefore follows that W_T in (A3) is also log-normally distributed. As a result, assuming CRRA utility it is straightforward to obtain an analytic expression for the value function corresponding to any static strategy as well as its derivatives. These terms can then be used to obtain an upper bound on the value function for the optimal dynamic trading strategy as described in section 4 and, in further detail, in HKW.

Appendix B: The GBH strategy and value function

We now expand the steps outlined in section 3.4 specializing to the dynamics of section 6.

B.1. Solving the SDE

The security price dynamics are as specified in (26) so that P_t is a three-dimensional price process and X_t is a scalar state variable process. The market-price-of-risk process, η_t , is a four-dimensional process satisfying the first three† equations of

$$\Sigma_P \eta_t = (\mu_0 + X_t \mu_1 - r \mathbf{1}). \quad (\text{B1})$$

The corresponding Q -Brownian motion satisfies

$$dB_t^P = dB_t^Q - \eta_t dt. \quad (\text{B2})$$

The first three components of η_t are uniquely‡ determined by (B1) and the fourth component, $\eta_t^{(4)}$, is unconstrained. Under any risk-neutral measure, Q , defined by (B1) and (B2), the security price processes satisfy $dP_t = P_t[r dt + \Sigma_P dB_t^Q]$. It immediately follows that

$\ln(P_T) \sim N(\mu_Q, \Sigma_Q)$ under any risk-neutral measure, Q , where

$$\begin{aligned} \mu_Q &= \ln P_0 + \left(r - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^\top) \right) T, \\ \Sigma_Q &= \Sigma_P I \Sigma_P^\top T. \end{aligned} \quad (\text{B3})$$

The state variable, X_t , is easily seen to satisfy

$$X_t = X_0 e^{-kt} + e^{-kt} \int_0^t e^{ks} \Sigma_X dB_s^P, \quad (\text{B4})$$

with

$$\begin{aligned} E_0[X_t] &= X_0 e^{-kt}, \\ \text{Var}(X_t) &= \frac{\Sigma_X \Sigma_X^\top}{2k} (1 - e^{-2kt}). \end{aligned} \quad (\text{B5})$$

Setting $Y_t := \ln P_t$, (B4) and a standard application of Itô's lemma then yield

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \left(\mu_0 - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^\top) + \mu_1 X_0 e^{-ks} \right) ds \\ &\quad + \int_0^t \Sigma_P dB_s^P + \int_0^t \mu_1 \frac{(1 - e^{k(s-t)})}{k} \Sigma_X dB_s^P. \end{aligned}$$

Under the true data generating measure, P , it therefore follows that $Y_T = \ln P_T \sim N(\mu_Y, \Sigma_Y)$, where

$$\mu_Y = \left(\mu_0 - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^\top) \right) T + \ln P_0 + \frac{\mu_1 X_0}{k} (1 - e^{-kT}), \quad (\text{B6})$$

$$\begin{aligned} \Sigma_Y &= (\Sigma_P \Sigma_P^\top) T + \mu_1 \mu_1^\top \Sigma_X \Sigma_X^\top \left(\frac{T}{k^2} + \frac{1 - e^{-2kT}}{2k^3} - \frac{2(1 - e^{-kT})}{k^3} \right) \\ &\quad + (\mu_1 (\Sigma_P \Sigma_X^\top)^\top + (\Sigma_P \Sigma_X^\top) \mu_1) \left(\frac{T}{k} - \frac{(1 - e^{-kT})}{k^2} \right). \end{aligned} \quad (\text{B7})$$

In particular, the PDF of P_T under P is multivariate log-normal and satisfies

$$\begin{aligned} f_{P_1, P_2, P_3}^P(b_1, b_2, b_3) &= \frac{1}{(\prod_{i=1}^3 b_i) (2\pi)^{3/2} |\Sigma_Y|^{1/2}} \\ &\quad \times \exp \left(-\frac{1}{2} (\ln b - \mu_Y)^\top \Sigma_Y^{-1} (\ln b - \mu_Y) \right). \end{aligned}$$

P_T has the same density under Q with the obvious replacement of μ_Y and Σ_Y with μ_Q and Σ_Q , respectively.

B.2. Computing the conditional state price density

A state price density (SPD) process, π_t , satisfies

$$\pi_t = e^{-rt} \frac{dQ}{dP} = e^{-rt} \exp \left(- \int_0^t \eta_s dB_s^P - \frac{1}{2} \int_0^t \|\eta_s\|^2 ds \right), \quad (\text{B8})$$

where η_s is any market price-of-risk process satisfying (B1). We need to compute the conditional state price density, π_t^b , where we condition on the terminal security

†Since there are three traded securities and the fourth ‘security’, X_t , is not traded.

‡See table 1 where we assume, without loss of generality, that $\eta_t^{(4)}$ does not influence the first three rows of Σ_P .

prices. In particular, we wish to solve for the time T conditional state price density

$$\pi_T^b = E_0[\pi_T | P_T^{(i)} = b_i, i = 1, \dots, N]. \quad (\text{B9})$$

It is worth mentioning that while there are infinitely many SPD processes, π_t , corresponding to each solution of (B1), it is easy to check if we can use any such process on the right-hand side of (B9) and still obtain the same conditional state price density, π_T^b . Using (B8) we can confirm it satisfies

$$\begin{aligned} \pi_T^b &= \frac{e^{-rT} f_{P_1, P_2, P_3}^Q(b_1, b_2, b_3)}{f_{P_1, P_2, P_3}^P(b_1, b_2, b_3)} \\ &= \frac{e^{-rT} |\Sigma_Y|^{1/2}}{|\Sigma_Q|^{1/2}} \exp\left(\frac{1}{2} ((\ln b - \mu_Y)^\top \Sigma_Y^{-1} (\ln b - \mu_Y) \right. \\ &\quad \left. - (\ln b - \mu_Q)^\top \Sigma_Q^{-1} (\ln b - \mu_Q))\right). \end{aligned}$$

B.3. Computing the optimal GBH wealth

We then use the static martingale approach to solve for the optimal GBH strategy assuming CRRA utility. In particular, we solve

$$V_0^{\text{gbh}} = \sup_{W_T} E_0 \left[\frac{W_T^{1-\gamma}}{1-\gamma} \right], \quad \text{subject to } E_0[\pi_T^b W_T] = W_0. \quad (\text{B10})$$

Because we use the conditional state price density in (B10), we do not need to explicitly impose[†] the $\sigma(P_T)$ -measurability of W_T as a constraint as this will be automatically satisfied. The problem in (B10) can be solved using standard static optimization techniques and we obtain

$$W_T^{\text{gbh}} = \frac{W_0 (\pi_T^b)^{-1/\gamma}}{E_0[(\pi_T^b)^{(\gamma-1)/\gamma}]}. \quad (\text{B11})$$

We call the trading strategy that replicates W_T^{gbh} the generalized buy-and-hold (GBH) trading strategy. We also immediately obtain

$$V_0^{\text{gbh}} = \frac{W_0^{1-\gamma}}{1-\gamma} (E_0[(\pi_T^b)^{(\gamma-1)/\gamma}])^\gamma. \quad (\text{B12})$$

While requiring some computation, it is straightforward to show that the expectation in (B12) is given by

$$\begin{aligned} E_0[(\pi_T^b)^{(\gamma-1)/\gamma}] &= \exp\left(-rT \frac{(\gamma-1)}{\gamma} - \frac{\delta}{2}\right) \\ &\quad \times \left(\frac{|\Sigma_Y|}{|\Sigma_Q|}\right)^{(\gamma-1)/(2\gamma)} \sqrt{\frac{|\Sigma|}{|\Sigma_Y|}}, \end{aligned} \quad (\text{B13})$$

where Σ_Q and Σ_Y are given in equations (B3) and (B7), respectively. The parameters δ and Σ are solutions to

$$\begin{aligned} \Sigma^{-1} &= (1-2a)\Sigma_Y^{-1} + 2a\Sigma_Q^{-1}, \\ \mu^\top \Sigma^{-1} &= (1-2a)\mu_Y^\top \Sigma_Y^{-1} + 2a\mu_Q^\top \Sigma_Q^{-1}, \\ \delta &= (1-2a)\mu_Y^\top \Sigma_Y^{-1} \mu_Y + 2a\mu_Q^\top \Sigma_Q^{-1} \mu_Q - \mu^\top \Sigma^{-1} \mu, \end{aligned} \quad (\text{B14})$$

where $a := (\gamma-1)/(2\gamma)$.

B.4. Determining the value function at all intermediate times

It is straightforward to generalize (B12), (B13) and (B14) to obtain

$$V_t^{\text{gbh}} = E_t \left[\frac{(W_T)^{1-\gamma}}{1-\gamma} \right] \quad (\text{B15})$$

$$\begin{aligned} &= \frac{W_0^{1-\gamma}}{(1-\gamma)} \exp\left(\frac{-\bar{\delta}_t + \delta(1-\gamma)}{2} + rT(1-\gamma)\right) \\ &\quad \times \left(\frac{|\bar{\Sigma}_t|}{|\Sigma_{tY}|}\right)^{1/2} \left(\frac{|\Sigma|}{|\Sigma_Q|}\right)^{(\gamma-1)/2}, \end{aligned} \quad (\text{B16})$$

where

$$\begin{aligned} \bar{\Sigma}_t^{-1} &= \Sigma_{tY}^{-1} - 2a\Sigma_Y^{-1} + 2a\Sigma_Q^{-1}, \\ \bar{\mu}_t^\top \bar{\Sigma}_t^{-1} &= \mu_{tY}^\top \Sigma_{tY}^{-1} - 2a\mu_Y^\top \Sigma_Y^{-1} + 2a\mu_Q^\top \Sigma_Q^{-1}, \\ \bar{\delta}_t &= \mu_{tY}^\top \Sigma_{tY}^{-1} \mu_{tY} - 2a\mu_Y^\top \Sigma_Y^{-1} \mu_Y + 2a\mu_Q^\top \Sigma_Q^{-1} \mu_Q \\ &\quad - \bar{\mu}_t^\top \bar{\Sigma}_t^{-1} \bar{\mu}_t, \end{aligned}$$

and where μ_{tY} and Σ_{tY} are the mean vector and covariance matrix, respectively, of $\ln P_T$ under P conditional on \mathcal{F}_t . In particular, $\ln P_T \sim N(\mu_{tY}, \Sigma_{tY})$ where μ_{tY} and Σ_{tY} are given by (B6) and (B7), respectively, but with T , P_0 and X_0 replaced by $(T-t)$, P_t and X_t , respectively.

B.5. Determining the replicating trading strategy

We now briefly describe how to obtain the replicating strategy for the optimal GBH wealth, W_T^{gbh} , given by (B11). The martingale property of a state-price density process implies

$$\pi_t W_t = E_t[\pi_T W_T] = \frac{W_0 E_t[\pi_T (\pi_T^b)^{-1/\gamma}]}{E_0[(\pi_T^b)^{(\gamma-1)/\gamma}]}. \quad (\text{B17})$$

Using (B1), (B8) and (B9) to substitute for π_T^b and π_T in (B17), we can evaluate the expectations in (B17) to obtain

$$W_t^{\text{gbh}} = \exp\left(rt + \frac{\delta}{2} - \frac{\delta_t}{2}\right) \left(\sqrt{\frac{|\Sigma_t|}{|\Sigma_{tQ}|}}\right) \left(\sqrt{\frac{|\Sigma|}{|\Sigma_Q|}}\right), \quad (\text{B18})$$

[†]Note that the more usual form of the constraint in (B10) is that $E_0[\pi_T W_T] = W_0$ where π_T is the time T value of a particular state price density process. If we use this constraint, then we need to explicitly impose the $\sigma(P_T)$ -measurability of W_T . In that case, however, while there are infinitely many such SPD processes, they all satisfy $E_0[\pi_T W_T] = E_0[E[\pi_T W_T | P_T]] = E_0[\pi_T^b W_T]$ when W_T is $\sigma(P_T)$ -measurable. We therefore obtain (B10) and no longer need to explicitly impose the $\sigma(P_T)$ -measurability of W_T .

where Σ_t and δ_t satisfy

$$\begin{aligned}\Sigma_t^{-1} &= \frac{\Sigma_Y^{-1}}{\gamma} - \frac{\Sigma_Q^{-1}}{\gamma} + \Sigma_{tQ}^{-1}, \\ \mu_t^\top \Sigma_t^{-1} &= \frac{\mu_Y^\top \Sigma_Y^{-1}}{\gamma} - \frac{\mu_Q^\top \Sigma_Q^{-1}}{\gamma} + \mu_{tQ}^\top \Sigma_{tQ}^{-1}, \\ \delta_t &= \frac{\mu_Y^\top \Sigma_Y^{-1} \mu_Y}{\gamma} - \frac{\mu_Q^\top \Sigma_Q^{-1} \mu_Q}{\gamma} + \mu_{tQ}^\top \Sigma_{tQ}^{-1} \mu_{tQ} - \mu_t^\top \Sigma_t^{-1} \mu_t.\end{aligned}\tag{B19}$$

The terms μ_{tQ} and Σ_{tQ} appearing in (B19) are the mean vector and variance-covariance matrix of $\ln P_T$ under Q , conditional on time t information. In particular, given time t information, we have $\ln P_T \sim N(\mu_{tQ}, \Sigma_{tQ})$ under any risk-neutral measure, Q , where

$$\begin{aligned}\mu_{tQ} &= \ln P_t + \left(r - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^\top)\right)(T-t), \\ \Sigma_{tQ} &= \Sigma_P I \Sigma_P^\top (T-t).\end{aligned}$$

We can then apply Itô's lemma to $W_t^{\text{gbh}} = f(t, P_t^{(1)}, P_t^{(2)}, P_t^{(3)})$ to obtain

$$\begin{aligned}dW_t &= \left[\frac{\partial f}{\partial t} + \sum_{j=1}^3 \sum_{i=1}^3 \frac{\partial^2 f}{\partial P_t^{(i)} \partial P_t^{(j)}} \right] dt + \frac{\partial f}{\partial P_t^{(1)}} dP_t^{(1)} \\ &\quad + \frac{\partial f}{\partial P_t^{(2)}} dP_t^{(2)} + \frac{\partial f}{\partial P_t^{(3)}} dP_t^{(3)}.\end{aligned}\tag{B20}$$

From (B18), we can see that W_t^{gbh} depends on P_t only through δ_t which in turn depends on P_t through μ_{tQ} and μ_t . Hence,

$$\frac{\partial W_t^{\text{gbh}}}{\partial P_t^{(i)}} = \frac{\partial f}{\partial P_t^{(i)}} = \frac{-W_t^{\text{gbh}}}{2} \frac{\partial \delta_t}{\partial P_t^{(i)}}.\tag{B21}$$

If $\theta_t^{(i)}$ is the proportion of W_t^{gbh} invested in the i th security at time t , then a standard argument using (1) and (2) implies that $\theta_t^{(i)}$ is the coefficient of $dP_t^{(i)}/P_t^{(i)}$ in the dynamics of dW_t/W_t . Using (B20) and (B21) we therefore obtain

$$\theta_t^{(i)} = \frac{-P_t^{(i)}}{2} \frac{\partial \delta_t}{\partial P_t^{(i)}}.\tag{B22}$$

B.6. Compute lower and upper bounds on true optimal value function

When the agent does not face dynamic trading constraints then the expected utility, V_0^{gbh} , can be attained by following the trading strategy outlined above in equation (B22). V_0^{gbh} , which we have obtained in closed form, is therefore a lower bound on the optimal value function associated with the true optimal dynamic trading strategy.

The same strategy and its associated value function process, V_t^{gbh} , can then be used to obtain an upper bound on the optimal value function. This is done by: (i) simulating the SDEs for the price processes, P_t , the state variable process, X_t , and the wealth process, W_t ; and (ii) following the algorithm outlined in HKW and summarized in section 4.

Appendix C: Optimizing the upper bound

In this appendix we determine the static strategies[†] that minimize the upper bound assuming the same[‡] price dynamics of section 6. We consider only the case of incomplete markets here and the minimum is taken over the set of all static strategies. Interestingly, we will see that the ‘optimal’ static strategy, θ^{static} , given by equation (8), does not minimize the upper bound. This of course is not too surprising. After all, and as we mentioned in section 5, this static strategy is not the true optimal static strategy.

As we saw in section 4, the value function (17) in the fictitious market provides an upper bound for the optimal value function of the original problem. In order to evaluate (17) we require the state-price density process, $\pi_t^{(v)}$, in the fictitious market and the corresponding market price-of-risk process, $\eta_t^{(v)}$. We compute $\eta_t^{(v)}$ by starting with the *candidate* market price-of-risk process, $\tilde{\eta}_t$, as determined by (23). When we employ a static trading strategy, θ , then $\tilde{\eta}_t$ can be determined analytically using proposition 5.1. In particular, we obtain

$$\tilde{\eta}_t = \gamma \Sigma_P^\top \theta - \frac{\Sigma_X^\top}{k} (1-\gamma)(1 - \exp(-k(T-t)))\theta^\top \mu_1.\tag{C1}$$

In the case of incomplete markets, i.e. our base case with no trading constraints, the risk-free rate process and $\eta_t^{(v)}$ actually satisfy[§]

$$\begin{aligned}r_t^{(v)} &= r, \\ \eta_t^{(v)} &= \begin{cases} \eta_{t,i}, & \text{for } i = 1, 2, 3, \\ \tilde{\eta}_{t,i}, & \text{for } i = 4, \end{cases}\end{aligned}\tag{C2}$$

where

$$\eta_t = \Sigma_P^{-1}(\mu_0 + X_t \mu_1 - r)$$

is the market price-of-risk process in the original problem. Note that $\eta_t^{(v)}$ is a (4×1) vector and that it depends upon the static strategy, θ , only through its fourth component. Recalling that $\theta_4 = 0$ and that the first three entries in the fourth column of Σ_P are zero, (C1) implies that this fourth component satisfies

$$\eta_{t,4}^{(v)} = -\frac{\Sigma_X^\top(4)}{k} (1-\gamma)(1 - \exp(-k(T-t)))\theta^\top \mu_1.\tag{C3}$$

[†]As we shall see, there are infinitely many strategies that minimize the upper bound.

[‡]Recall from section 6 that the drift of the asset returns, $\mu(X_t)$, satisfies $\mu(X_t) = \mu_0 + X_t \mu_1$ where μ_0 and μ_1 are 4×1 vectors. In addition, θ is also a 4×1 vector with the fourth component set to 0.

[§]Recall that $\tilde{\eta}_t$ is not necessarily an element of the admissible set \mathbf{D} as defined in section 4. We need to solve a simple quadratic optimization problem with linear constraints to obtain the admissible process, $\eta_t^{(v)}$, of equation (C2). See HKW for further details.

The upper bound given by (17) then satisfies

$$\begin{aligned} UB &= \frac{W_0^{1-\gamma}}{1-\gamma} E_0[\pi_T^{(v)(\gamma-1)/\gamma}]^\gamma \\ &= \frac{W_0^{1-\gamma}}{1-\gamma} E_0 \left[\exp \left(\frac{\gamma-1}{\gamma} \left(-rT - \frac{1}{2} \sum_{i=1}^4 \int_0^T (\eta_{t,i}^{(v)})^2 dt \right. \right. \right. \\ &\quad \left. \left. \left. - \sum_{i=1}^4 \int_0^T \eta_{t,i}^{(v)} dB_t^{(i)} \right) \right) \right]^\gamma. \end{aligned} \quad (C4)$$

The first-order conditions for minimizing the upper bound with respect to θ_j can be expressed as

$$\begin{aligned} E_0 \left[\frac{\partial \pi_T^{(v)(\gamma-1)/\gamma}}{\partial \theta_j} \right] &= \frac{(\gamma-1)}{\gamma} E_0 \left[\pi_T^{(v)(\gamma-1)/\gamma} \left(\int_0^T \eta_{t,4}^{(v)} \frac{\partial \eta_{t,4}^{(v)}}{\partial \theta_j} dt + \int_0^T \frac{\partial \eta_{t,4}^{(v)}}{\partial \theta_j} dB_t^{(4)} \right) \right] \\ &= 0, \end{aligned} \quad (C5)$$

for $j = 1, 2, 3$. This condition simplifies to

$$\begin{aligned} E_0 \left[\frac{\partial \pi_T^{(v)(\gamma-1)/\gamma}}{\partial \theta_j} \right] &= E_0 \left[\pi_T^{(v)(\gamma-1)/\gamma} \left(\int_0^T \frac{\Sigma_X^\top(4)}{k} (1-\gamma) \right. \right. \\ &\quad \times (1 - \exp(-k(T-t)))^2 \theta^\top \mu_1 dt \\ &\quad \left. \left. - \int_0^T (1 - \exp(-k(T-t))) dB_t^{(4)} \right) \right] = 0, \end{aligned} \quad (C6)$$

where the right-hand side is independent of j . This implies that the three first-order conditions are therefore identical

and can be reexpressed as

$$\theta^\top \mu_1 = \frac{E_0 \left[\pi_T^{(v)(\gamma-1)/\gamma} \int_0^T (1 - \exp(-k(T-t))) dB_t^{(4)} \right]}{\left\{ \left(T + \frac{1 - \exp(-2kT)}{2k} - \frac{2(1 - \exp(-kT))}{k} \right) \right.} \\ \left. \left(\frac{\Sigma_X^\top(4)(1-\gamma)}{k} \right) E_0[\pi_T^{(v)(\gamma-1)/\gamma}] \right\}}. \quad (C7)$$

Any static strategy that minimizes the upper bound must satisfy (C7). As this is a single linear equation in three unknowns we conclude that there are infinitely many static strategies that minimize, over the class of static strategies, the upper bound on the optimal value function. For example, it is possible to construct a static strategy that invests in only one of the risky assets and that satisfies (C7). Such a strategy would minimize the upper bound, though it could, in general, generate a poor lower bound. Another strategy that minimizes the upper bound is a suitably scaled version of θ^{static} , where the scaling is required so that (C7) is satisfied.

In our numerical experiments it was often necessary, for example, to scale θ^{static} by an additional 10% before (C7) was satisfied. However, it is worth mentioning that despite the substantial scaling required to satisfy (C7), the upper bound generated by θ^{static} was typically within one or two basis points per annum of the optimal upper bound. Moreover, we never saw this difference exceeding 10 basis points. We will not report any explicit results comparing the optimal upper bound with the upper bound generated by θ^{static} as the two bounds are close in practice.