Technical Note: Multi-Market Cournot Equilibria with Heterogeneous Resource-Constrained Firms

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Abstract

We study Cournot competition among firms in a multi-market framework where each of the firms face different budget / capacity constraints. We assume independent linear inverse demand functions for each market, and completely characterize the resulting unique equilibrium. Specifically, we introduce the notions of *augmented* and *cutoff* budgets for firms and markets, respectively. We show, for example, that firm i operates in market j if and only if firm i's augmented budget is greater than market j's cutoff budget. We also study the properties of the equilibrium as a function of the number of firms N while keeping the aggregate budget fixed. In a numerical study, we show that increasing N increases the total output across all markets although this monotonicity can fail to hold at the individual market level. Similarly, we show that that while the firms' cumulative payoff decreases in N, the consumer surplus and social surplus increase in N.

Keywords: Cournot competition, non-cooperative games, heterogeneous products, heterogeneous firms, capacity constraints.

1 Introduction

In this note we study a model of Cournot competition (Cournot, 1838) in a multi-firm multi-market environment in which each of the firms face different budget or capacity constraints. (We will use budget and capacity, and market and product interchangeably throughout.) The firms compete by choosing what quantities to produce and prices are then determined so as to clear the market[1]. In this setting, we completely characterize the resulting unique Cournot market equilibrium and investigate the effects that these capacity constraints have on the equilibrium outcome, including total output across markets, firms' payoffs and social welfare.

From a modeling standpoint, we adopt one of the most commonly used frameworks in the Cournot literature with independent linear inverse demand functions for each market and linear cost functions. At the same time we consider a fundamental – and curiously unexplored – extension of this standard model by explicitly incorporating firm-level budget (or capacity) constraints that limit the production decisions of the firms and couple the equilibrium outcome across markets. This novel feature of our model not only brings our analysis closer to reality but also produces new insights on the nature of the resulting Cournot equilibrium. For instance, it is no longer true that every firm operates in every market or that there is a positive output on every market in equilibrium. For example, this market concentration is commonly observed in the airline industry where competition and available seat capacity constraints force airlines to operate only on high-demand routes to ensure profitability (*e.g.*, Flores-Fillol, 2009) leading to decreased connectivity or frequency on less profitable routes (*e.g.*, Mazzeo, 2003, Gil and Kim, 2021). Additionally, by explicitly incorporating capacity and budget constraints into the market equilibrium, we provide a more nuanced analysis of the competitive pressures influencing firms' incentives for mergers and acquisitions (see, for example, Werden et al. (1991) and Das (2019) for a related discussion in the context of airline merges).

To formalize these ideas we introduce the notions of *augmented* and *cutoff* budgets for firms and markets, respectively. Loosely speaking, the augmented budget of a firm measures its level of competitiveness vis-à vis the other firms and determines in which market the firm will operate, i.e., where it will allocate a positive amount of its budget. On the other hand, the cutoff budget of a market is the minimum amount required for investing in all other markets to ensure that this particular market will operate in equilibrium. (We refer the reader to Definition 3.1 for precise mathematical definitions and Appendix A for additional intuition regarding these concepts.) To the best of our knowledge these are new concepts that have not been previously discussed in the literature and they allow us to capture in a parsimonious way the intricate interactions among firms and markets. In particular, we show that in equilibrium firm i operates in market j if and only if firm i's augmented budget is greater than market j's cutoff budget. We also derive monotonicity results on the equilibrium market outputs with respect to both firm size (as measured by budget or capacity) and market size. We also study the properties of the equilibrium as a function of the number of firms N while keeping the aggregate budget fixed. In a numerical study, we show that increasing N increases the total output across all markets although this monotonicity can fail to hold at the individual market level. In other words, increasing competition can lead to some markets shutting down in equilibrium. Similarly, we show that while the firms' cumulative payoff

[1]This form of competition is more suitable for industries where production (or production capacities) must be planned in advance, e.g. semiconductor manufacturing, electricity markets.

decreases in N, the consumer surplus and social surplus increase in N. While not the main focus of this note, we also consider non-linear inverse-demand functions. We show multiple equilibria are possible when they are piecewise-linear and provide a characterization of any equilibrium when they are concave and sufficiently smooth.

The literature on oligopolistic competition and Cournot competition in particular, is vast and we cannot do justice to it here. Instead we refer the reader to the textbooks Okuguchi and Szidarovszky (1999) and Vives (2001). They provide detailed treatments covering such questions as existence, uniqueness, and stability of equilibria, as well as properties of the equilibria and how they relate to market structures, etc. Some work from the operations literature is particularly relevant, however, and concerns settings in which firms compete across multiple products and markets[2]. For example, the firms in Kluberg and Perakis (2012) produce multiple differentiated products and face asymmetric production constraints. Their model is more general in that the linear cost functions vary with firm and they allow more general affine inverse demand functions that allows for cross-market dependence in such a way that the products are gross substitutes for each other. However, most of their analysis assumes a single product per firm and they don't explicitly characterize the equilibrium. In contrast, in our model each of the firms can produce each product but the product markets are independent in the sense that a change in the quantity produced for one product has no impact on the clearing price for other products. Nonetheless the product markets are all coupled via the capacity constraints and we explicitly characterize the unique equilibrium. Motivated by problems in communications networks and airports, Perakis and Sun (2014) consider Cournot competition in service industries where the firms compete for users who are sensitive to both prices and congestion. They consider congestion with and without spillover costs and they quantify the efficiency of an unregulated oligopoly w.r.t the optimal social welfare. Other papers that also model competition across multiple markets include Allon and Federgruen (2009), Perakis and Roels (2007) and Federgruen and Hu (2015) but the form of competition in these papers is not Cournot and the firms are not budget / capacity constrained.

More recently, there has been work on Cournot competition across markets with a network structure. For example, Bimpikis et al. (2019) use a bipartite graph to model which subset of markets (of a homogeneous good) a firm can supply to. They characterize the unique Cournot equilibrium under a linear inverse demand function and relate it to supply paths in the underlying network structure. Related work includes Abolhassani et al. (2014) who study a more general version of Cournot competition in networked markets and Cai et al. (2019) who consider a similar problem but focus on the role of a market-maker in determining the resulting Cournot equilibrium and whether or not whether or not there is a unique equilibrium. Motivated by the operations of online platforms, Lin et al. (2017) contrast the market efficiency of open access versus discriminatory access platforms using a networked Cournot competition model. An important difference between our work and this existing literature on networked Cournot competition is that we explicitly impose capacity constraints on the quantities that each firm can supply across markets.

The remainder of this note is organized as follows. In Section 2 we formulate the Cournot equilibrium problem. Then in Section 3 we introduce the notions of augmented and cutoff budgets before

[2]We note that there exists an extensive operations management literature devoted to the study of Cournot equilibrium in a single market under various operational characteristics on the firms production function. Some representative examples include Deo and Corbett (2009), Downward et al. (2010) Jansen and Özaltin (2017), and references therein.

using them to fully characterize the equilibrium. In Section 4 we use our results from Section 3 to study the sensitivity of the equilibrium (aggregate output, consumer surplus, social welfare etc.) to the total number of firms whilst keeping the aggregate budget fixed. We conclude in Section 5 where we also outline some directions for future research. Some intuition regarding the concepts of augmented and cutoff budgets are provided in Appendix A. Proofs for the various results are provided in Appendix B while Appendix C considers how the analysis might extend to non-linear inverse-demand functions.

2 Problem Formulation

We consider a setting where N firms engage in Cournot competition in M different markets. We assume that firm $i \in [N]$ is budget constrained and can spend no more than $B_i \geq 0$ dollars in total. (For a positive integer k, we let $[k] := \{1, 2, \ldots, k\}$). Firm i competes in market $j \in [M]$ by allocating an amount $x_{ij} \geq 0$ of its budget subject to the budget constraint $\sum_{j=1}^{M} x_{ij} \leq B_i$.

The profits that firm i makes on market j depend on its budget allocation x_{ij} as well as on the cumulative budget spent by all firms in the market. Specifically, we assume that firm i's profit in market j is equal to $r_j(X_j)\, x_{ij}$, where $X_j := \sum_{i=1}^{N} x_{ij}$ is the total budget allocated to market j by all N firms and the function $r_j(\cdot)$ models market's j return per unit of investment. Given a budget allocation $\{x_{ij}\}_{j\in[M]}$, firm i collects a net profit $\sum_{j=1}^{M} r_j(X_j)\, x_{ij}$. We will assume a linear demand model so that $r_j(x) = R_j - x/\beta_j$ for two positive parameters R_j and β_j for all $j \in [M]$. Without loss of generality, we rank the markets so that $0 \leq R_1 \leq R_2 \leq \cdots \leq R_M$ and we will refer to the pair (R_j, β_j) as the j^{th} market. For future reference, we will denote by $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ the collection of markets, and by $\mathbf{B} = (B_1, \ldots, B_N)$ the vector of budgets. Again without loss of generality, we assume the elements in $\mathbf{B}$ have been ordered so that

$$B_1 \geq B_2 \geq \cdots \geq B_N > 0. \tag{1}$$

Let $X_{ij-} := X_j - x_{ij}$ be the total budget allocated to market j by all firms except firm i and define $\mathbf{X}_{i-} := (X_{i1-} \ \ldots \ X_{iM-})$. For a given value $\mathbf{X}_{i-}$, firm's i best-response budget-allocation strategy $\{x^*_{ij}(\mathbf{X}_{i-})\}_{j\in[M]}$ solves the optimization problem:

$$\Pi_i(\mathbf{X}_{i-}) := \max_{x_{ij}\geq 0} \sum_{j=1}^{M} r_j(X_{ij-} + x_{ij})\, x_{ij} \qquad \text{subject to} \qquad \sum_{j=1}^{M} x_{ij} \leq B_i. \tag{2}$$

Definition 2.1 (Cournot Equilibrium) *Consider a collection of markets $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ and a set of N firms with budgets $\mathbf{B} = (B_1, \ldots, B_N)$. A* Cournot equilibrium *is a set of budget allocations $\{x^*_{ij}\}_{j\in[M]}$ for $i \in [N]$ chosen by the firms so that:*

(i) *$\{x^*_{ij}\}_{j\in[M]}$ solves the optimization problem (2) for $i \in [N]$.*

(ii) *They satisfy the fixed-point condition:*

$$X_{ij-} = \sum_{k\neq i} x^*_{kj}(\mathbf{X}_{k-}) \qquad \textit{for all } i \in [N] \quad \textit{and} \quad j \in [M]. \tag{3}$$

Before proceeding to analyze the Cournot game and solving for its equilibrium, we describe several applications that can be modelled using this framework. In each case the players are assumed to be in Cournot competition.

1. Production Scheduling: There are N manufacturers who can produce M different products. The total production capacity of the i^{th} manufacturer is B_i and it allocates x_{ij} units of this capacity to the production of product j. Other settings are also possible. For example, we could consider the settings where B_i represents a *space* budget.

2. Airline Revenue Management: There are N airlines and a total of M routes under consideration. The i^{th} airline has a total of B_i passenger seats to allocate to the M routes. Then x_{ij} represents the number of seats allocated by the i^{th} airline to the j^{th} route (i.e., partition booking limits). Market competition in the airline industry has been studied in Brander and Zhang (1990), Kim and Singal (1993), Hu et al. (2012), Alves and Forte (2015) and references therein.

3. Intertemporal Competition with Exhaustible Resources: There are N firms each endowed with a finite amount of an exhaustible resource (e.g., crude oil or minerals). Firms compete by deciding the amount of resource (i.e., budget) that they want to put on the market over time. For example, if there are M time periods then we can view them as constituting M independent markets where the natural ordering of time is irrelevant. This setting is then essentially identical to the Production Scheduling setting above. More generally, however, we could consider a sequential game where the ordering of time certainly does matter. Rather than pursuing this route (where additional structure might be required to handle the issue of multiple equilibria), we merely note that the static Cournot equilibrium of this note could serve as a building block for analyzing the sequential version of the game. We note that others (e.g., Maskin and Tirole, 1987, Ludkovski and Sircar, 2012) have also studied dynamic Cournot games with limited production resources.

4. Financial Hedging in Supply Chains: There are N retailers who purchase a *single* product from a producer at time $t = 0$. There are M possible states of nature at time $t = 1$, each of which occurs with probability p_j, for $j = 1, \ldots, M$. The producer allows the ordering quantity to be state dependent with q_{ij} denoting the quantity purchased by retailer i in state j. The producer charges v_j per unit ordered in state j. Let A_j and Q_j denote the market size and total quantity ordered by all N retailers, respectively, in state j. The i^{th} retailer's objective is then given by $\sum_{j=1}^{M} ((A_j - Q)q_{ij} - q_{ij}v_j)\, p_j$. A so-called *complete financial market* assumption allows the retailer to allocate (via, for example, a dynamic financial trading strategy) the budget B across the M states. This means the i^{th} retailer only has to satisfy the budget constraint *in expectation*, i.e. she must satisfy $\sum_{j=1}^{M} p_j v_j q_{ij} \le B_i$. Letting $x_{ij} := p_j q_{ij}$, the i^{th} retailer's problem becomes $\max_{x_{ij}} \sum_{j=1}^{M} ((A_j - Q)x_{ij} - x_{ij}v_j)$ subject to $\sum_{j=1}^{M} v_j x_{ij} \le B_i$. The v_j term in the objective can be absorbed into the A_j term after another change of variables and so the retailer's problem can be cast as in (2). This application follows[3] Caldentey and Haugh (2021) who only consider the symmetric case where the budgets B_i are identical.

[3] Caldentey and Haugh (2021) also allow the producer to choose the v_j's as a Stackelberg leader. In addition, the retailers are also allowed to access costly debt markets to help them circumvent their budget constraints.

Further Discussion of Model Assumptions: One aspect of our model that deserves further discussion is our assumption that production costs are homogeneous across firms and markets. While we have restricted ourselves to this case for mathematical tractability, it is worth noting that we can partially relax this assumption and replace firm i's budget constraint in (2) with $\sum_i w_{ij}x_{ij} \leq B_i$, where the weights w_{ij} admit the decomposition $w_{ij} = \theta_i \eta_j$ for all $i \in [N]$ and $j \in [M]$ with $\theta_i > 0$ and $\eta_j > 0$. We interpret θ_i as a measure of the production efficiency of firm i and η_j as a measure of the costs of serving market j. For example, with x_{ij} denoting the number of units that firm i sells in market j, we can think of θ_i as firm i's per-unit production cost and η_j as a cost factor that captures transportation and other commercialization fees that firms incur when serving market j[4].

In this heterogeneous cost setting, firm i's best-response problem becomes:

$$\max_{x_{ij}\geq 0} \sum_{j=1}^{M} \left[R_j - \frac{X_{ij-} + x_{ij}}{\beta_j}\right] x_{ij} \qquad \text{subject to} \qquad \sum_{j=1}^{M} \theta_i\, \eta_j\, x_{ij} \leq B_i.$$

We can reduce this new formulation back to our base model in (2) with homogeneous costs by introducing the change of variables

$$\widehat{B}_i = \frac{B_i}{\theta_i}, \qquad \widehat{x}_{ij} = \eta_j\, x_{ij}, \quad \widehat{\beta}_j = \eta_j^2\, \beta_j, \quad \text{and} \quad \widehat{R}_j = \frac{R_j}{\eta_j}$$

and firm i's best-response problem becomes

$$\max_{\widehat{x}_{ij}\geq 0} \sum_{j=1}^{M} \left[\widehat{R}_j - \frac{\widehat{X}_{ij-} + \widehat{x}_{ij}}{\widehat{\beta}_j}\right] \widehat{x}_{ij} \qquad \text{subject to} \qquad \sum_{j=1}^{M} \widehat{x}_{ij} \leq \widehat{B}_i.$$

Admittedly, the decomposition $w_{ij} = \theta_i\, \eta_j$ is somewhat restrictive. For example, it does not allow for the modelling of asymmetric competitive advantages that some firms can have in some markets or the imposition of a network structure. Nonetheless, it does provide a considerable degree of flexibility and might provide a useful approximation in many realistic settings.

Another aspect of our model that deserves further discussion is our choice of linear inverse demand functions which are quite common in the oligopoly literature. See, for example, Levitan and Shubik (1972), Singh and Vives (1984), Szidarovszky and Okuguchi (1988), Bernstein and Federgruen (2004), Yao et al. (2008), Farahat and Perakis (2011), Kluberg and Perakis (2012) and Federgruen and Hu (2015) as well as the textbook by Vives (2001). Establishing existence and uniqueness of a Cournot equilibrium using these demand functions is often quite straightforward using standard techniques. Indeed establishing existence and uniqueness for more general concave inverse demand functions can often be tackled using the concave games framework of Rosen (1965).

Our choice of linear inverse demand functions is also restricted by the assumption of independent markets so that a change in the quantity produced for one market has no impact on the clearing price in other markets. A second restriction of our model is that we assume all of the firms have access to all of the markets. This contrasts, for example, with the network Cournot literature discussed in Section 1 where a bipartite graph (firms on one side and markets on the other) is used to model which firms can access which markets. However, modelling a network structure can be handled

[4]With this interpretation we should have $\eta_j \geq 1$.

as a special case of general heterogeneous costs. (If firm i cannot access market j, for example, then this can be captured by setting a sufficiently high cost w_{ij}.) A third restriction of our model relates to the assumption of linear production costs but this is a very common assumption in the literature. Imposing these restrictions, however, allow us to fully characterize the unique Cournot equilibrium despite the presence of capacity constraints. Indeed, to the best of our knowledge, we are the first to produce such a characterization in multi-market oligopoly games with capacity constraints. Moreover, our characterization independently establishes existence and uniqueness by construction.

3 The Cournot Equilibrium

In this section we provide a complete derivation of the Cournot equilibrium. One of the main difficulties in finding a Cournot equilibrium in a multi-market setting when firms are budget constrained is that it is generally not the case that every firm operates in every market in equilibrium. For example, a low budget firm might be better off not competing in a market with a small market size to avoid an inefficient allocation of its budget.

A key step in the derivation is that we can provide a relatively simple characterization of the markets in which a firm operates by introducing the concepts of '*augmented*' and '*cutoff*' budgets.

Definition 3.1 (Augmented and Cutoff Budgets)

(i) *For a vector of firms' budgets* $\mathbf{B} = (B_1, \ldots, B_N)$, *the* augmented budget *associated to firm i is given by*

$$\mathcal{B}_i := i\, B_i + \sum_{k=i}^{N} B_k, \qquad \textit{for all } i \in [N]. \tag{4}$$

(ii) *For a set of linear markets* $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j)\colon j \in [M]\}$, *the* cutoff budget *of market j is given by*

$$\mathbb{B}_j := \sum_{k=j}^{M} \beta_k\, (R_k - R_j), \qquad \textit{for all } j \in [M]. \tag{5}$$

We also define $\mathbb{B}_0 := \sum_{k=1}^{M} \beta_k\, R_k$.

Recall that the firms have been ordered so that $B_1 \geq B_2 \geq \ldots \geq B_N$. It follows that the sequence $\{\mathcal{B}_i\}$ is also non-increasing in i. It is also worth noting that $\mathcal{B}_i$ does not depend on $\{B_1, \ldots, B_{i-1}\}$, i.e., on the $(i-1)$ highest budgets. Similarly, since the markets have been ordered so that $0 \leq R_1 \leq \cdots \leq R_M$, it follows that the $\{\mathbb{B}_j\}$'s are non-increasing in j and the value of $\mathbb{B}_j$ is independent of the characteristics of markets $\{1, 2, \ldots, j-1\}$. As we shall see, the significance of the augmented and cutoff budgets is that in equilibrium, firm i operates in market j if and only if $\mathcal{B}_i > \mathbb{B}_j$. Some intuition regarding the concepts of augmented and cutoff budgets is provided in Appendix A. Before we can formally state our equilibrium result, one further definition is needed.

Definition 3.2 *For a given collection of linear markets* $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$*, we define the function* $H(x)$ *according to*

$$H(x) := \inf\left\{z \geq 0 \ \ \textit{such that} \ \ \sum_{j=1}^{M} \beta_j \left(R_j - z\right)^+ \leq x\right\}, \quad \textit{for any } x \geq 0.$$

We note that $H(\cdot)$ is a continuous, non-increasing and piece-wise linear function that satisfies $H(\mathbb{B}_j) = R_j$ for all $j \in [M]$. We are now ready to state our main result.

Theorem 1 (Cournot Equilibrium)

Given a collection of linear markets $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ *and a set of firms with budgets* $\mathbf{B} = (B_1, \ldots, B_N)$ *satisfying the conditions in (1), there is a unique Cournot equilibrium* $\{x^*_{ij}\}$ *that satisfies*

$$x^*_{ij} = \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)}\right]^+, \tag{6}$$

where the $\{\mathcal{B}_i\}$*'s are the firms' augmented budgets defined in equation (4) and*

$$n^*_j := \max\left\{i \in [N] \ \textit{such that} \ \ \mathcal{B}_i > \mathbb{B}_j\right\} \tag{7}$$

is the number of firms operating in market j *where the* $\{\mathbb{B}_j\}$*'s are the markets' cutoff budgets defined in (5).*

It is interesting to note that (6) implies that firm i's allocations x^*_{ij} depend on the vector of budgets $\mathbf{B} = (B_1, \ldots, B_N)$ only through the values of $(\mathcal{B}_i, \mathcal{B}_{i+1}, \ldots, \mathcal{B}_N)$, which according to (4) are all independent of the value of the highest $i-1$ budgets $(B_1, B_2, \ldots, B_{i-1})$. At the same time, $(\mathcal{B}_i, \mathcal{B}_{i+1}, \ldots, \mathcal{B}_N)$ do depend on i, i.e. on the *number* of firms that have a budget greater than or equal to firm i's budget. In other words, the allocation decisions of a firm are unaffected by the size (but not the number) of larger firms.

An interesting special case is the symmetric case where we have $B_i = B$ for all i. In this case $\mathcal{B}_i = (N+1)B$ and the expression for x^*_{ij} in (6) also simplifies considerably. In particular, since $H(\mathcal{B}_k) = H((N+1)B)$ is independent of k, both summations in (6) can be expressed as telescoping sums and considerable simplification occurs. This leads to the following corollary.

Corollary 1 (Symmetric Firms) *Suppose the* N *firms are identical with* $B_i = B$ *for all* $i \in [N]$*. Then in equilibrium*

$$x^*_{ij} \quad = \quad \frac{\beta_j (R_j - H((N+1)B))^+}{1+N}. \tag{8}$$

The following proposition builds on Theorem 1. It (i) establishes monotonicity properties of the equilibrium ordering quantities and (ii) highlights the central role played by the augmented and cutoff budgets in the equilibrium.

Proposition 1 *Given a collection of linear markets* $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) : j \in [M]\}$ *and a set of firms with budgets* $\mathbf{B} = (B_1, \ldots, B_N)$ *ordered as in (1), the Cournot equilibrium satisfies:*

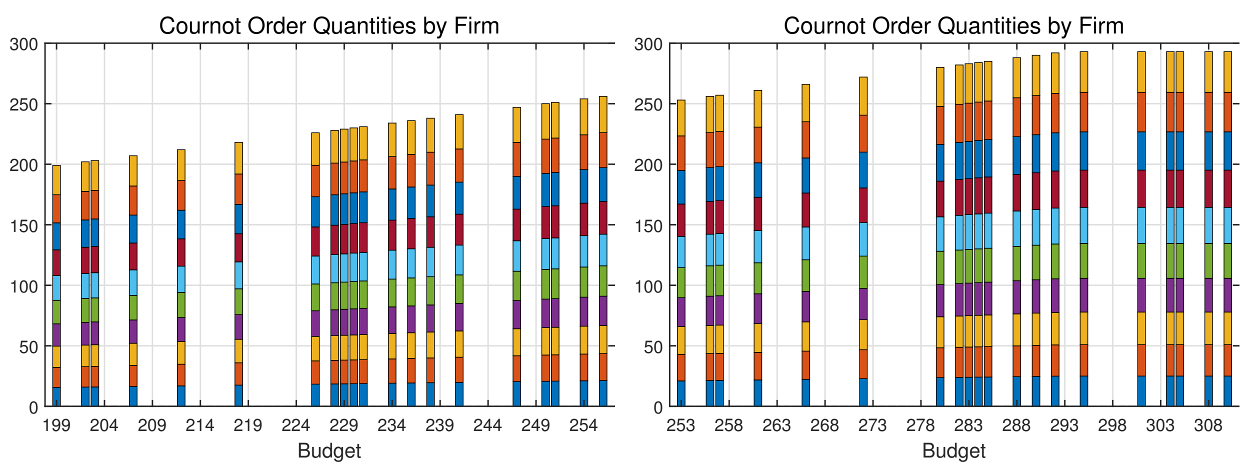
(i) $x_{ij}^* > 0$*, i.e. firm* i *operates in market* j*, if and only if* $\mathcal{B}_i > \mathbb{B}_j$*. In particular,* $x_{iM}^* > 0$ *for all* $i \in [N]$ *since* $\mathbb{B}_M = 0$ *(see equation 5). In addition,* x_{ij}^* *is non-increasing in* i *and* x_{ij}^*/β_j *is non-decreasing in* j*.*

(ii) $X_j^* = \sum_{i=1}^N x_{ij}^* > 0$*, i.e. market* j *is active, if and only if* $\mathcal{B}_1 > \mathbb{B}_j$*. Furthermore,*

$$X_j^* = \beta_j \left[\frac{n_j^* R_j}{1 + n_j^*} - \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} \right].$$

(iii) $\sum_{j=1}^M x_{ij}^* = B_i$*, i.e. firm* i*'s budget constraint is binding, if and only if* $\mathcal{B}_i \leq \mathbb{B}_0$*.*

Figure 1 depicts the Cournot equilibrium ordering quantities for $M = 10$ markets and $N = 20$ firms. In panel (a) all budgets are binding whereas in panel (b) only the smallest fourteen budgets are binding. Different bars correspond to different firms and different colors correspond to different markets. Moreover, we assumed the β_j's were constant w.r.t j and so the monotonicity w.r.t both i and j from part (i) of Proposition 1 is evident. In particular, (i) we see firms increase their allocations to each market as their budgets increase and (ii) for a fixed budget, i.e. firm, we see the allocation to market j increases in j. (For a fixed budget, the j^{th} segment in the bar corresponds to the j^{th} market.)



(a) All budgets binding.

(b) Smallest fourteen budgets binding.

Figure 1: Cournot equilibrium ordering quantities for $M = 10$ markets and $N = 20$ firms. In Figure 1(a) all budgets are binding whereas in Figure 1(b) only the first fourteen firms have their budgets binding. Other details are provided in the main text.

4 Sensitivity Analysis on the Number of Firms

We now investigate how the outcome of the Cournot equilibrium in Theorem 1 changes as a function of the total number of firms N. To this end, let us denote by $B_i(N)$ and $X_j^*(N)$ the budget of firm

$i \in [N]$ and the equilibrium output in market $j \in [M]$, respectively, when there are N competing firms. In order to have a meaningful comparison of how $X_j^*(N)$ varies with N, we assume that the cumulative budget $B_{\text{C}} := \sum_{i=1}^{N} B_i(N)$ is constant and therefore independent of N throughout this section.

So as to capture different budget distributions across the N firms, we will assume the budget of firm i (when there are N firms in the market) is equal to

$$B_i^{\text{N}} = \left[f\left(\frac{i-1}{N}\right) - f\left(\frac{i}{N}\right)\right] B_{\text{C}}, \qquad i \in [N], \tag{9}$$

for some given non-decreasing and convex function $f : [0,1] \to [0,1]$ with $f(0) = 1$ and $f(1) = 0$. The monotonicity and convexity of f guarantee that $B_1^{\text{N}} \geq B_2^{\text{N}} \geq \cdots \geq B_N^{\text{N}}$ as assumed in (1).

Combining equations (9) and (4) we see that the augmented budget of firm i equals

$$\mathcal{B}_i^{\text{N}} = \left[(i+1)\, f\left(\frac{i-1}{N}\right) - i\, f\left(\frac{i}{N}\right)\right] B_{\text{C}}.$$

As a concrete example, consider the exponential family of functions defined as

$$\mathcal{F} := \left\{ f_\alpha(x) = (e^{-\alpha\, x} - e^{-\alpha})/(1 - e^{-\alpha}) \colon \alpha \geq 0 \right\}.$$

The value of α controls the degree of heterogeneity in the distribution of B_{C} across the firms. On one extreme we have $\lim_{\alpha \downarrow 0} B_i^{\alpha}(N) = B_{\text{C}}/N$ so that the cumulative budget is uniformly distributed across firms as α approaches 0. On the other extreme we have $\lim_{\alpha \to \infty} B_i^{\alpha} = B_{\text{C}}\, \mathbb{1}(i = 1)$ so that the cumulative budget is allocated entirely to firm 1 in the limit as α goes to infinity.

The six panels in Figure 2 depict aggregate Cournot equilibrium outputs as a function of the number of firms (N) when there are $M = 10$ markets for each value of $\alpha \in \{0, 5, 10, \infty\}$. For the three panels on the top row we set $B_{\text{C}} = 0.9\, \mathbb{B}_0$ while for the three panels in the bottom row we set $B_{\text{C}} = 1.1\, \mathbb{B}_0$. (Recall from part (iii) of Proposition 1 that firm i's budget is binding if and only if $\mathcal{B}_i \leq \mathbb{B}_0$. This means the "average" firm has a binding budget constraint in the top row of panels and a non-binding budget constraint in the bottom row of panels.) The two panels on the left display the total output across all M markets, i.e. $\sum_{j=1}^{M} X_j^*$. The middle and right panels depict the values of X_1^* and X_M^*, respectively.

We see that increasing the value of N, i.e. increasing competition among firms, increases the total output across all markets. (When $\alpha = \infty$ there is effectively only 1 firm as explained above so in this case the total output is independent of N.) The effect of competition on the output of a particular market can be positive or negative, however. For example, when $B_{\text{C}} = 0.9\, \mathbb{B}_0$, the total output X_1^* in market 1 can decrease and even become zero as N increases. This is most easily explained by recognizing that there are two effects at play. The first effect is the competition effect mentioned above whereby increasing N increases the total output in the markets. For the second effect, we first note that an increase in N causes each retailer's individual budget to decrease since B_{C} is held fixed. As N increases this forces the firms to concentrate more of their ever scarce resources on the more profitable markets and less on the less profitable ones. Because of our indexing, market 1 is in fact the least attractive market and the middle sub-figure in the upper panel of Figure 2 suggests that this second effect begins to dominate beyond values of $N = 9$.

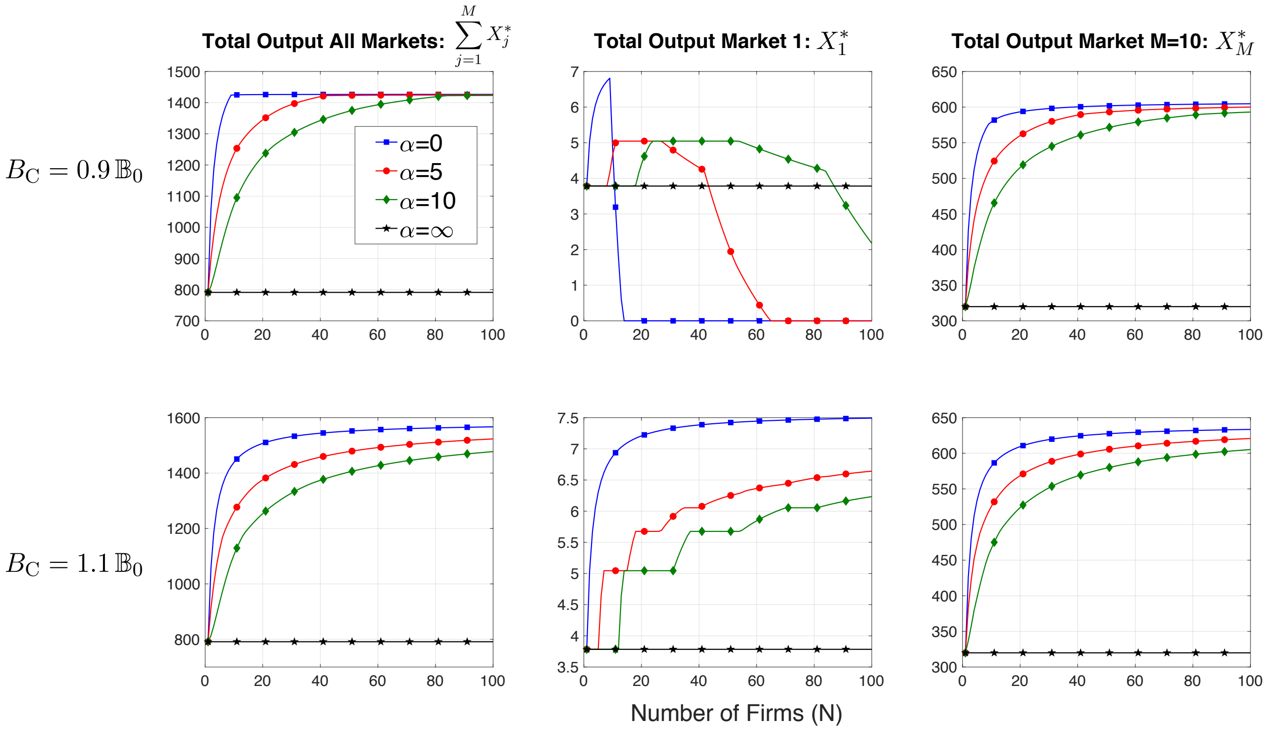


Figure 2: Aggregate Cournot equilibrium outputs for the case in which $M = 10$ for four distributions of budgets corresponding to $\alpha \in \{0, 5, 10, \infty\}$ as function of the number of firms (N).

We also see that the more uniformly distributed the total budget B_C is among the N firms (corresponding to smaller values of α), the larger the cumulative output $\sum_{j=1}^{M} X_j^*$. However, this aggregate monotonicity does not hold across each of the individual markets. This is demonstrated by the totals for X_1^* in the middle panel of Figure 2 when $B_C = 0.9\,\mathbb{B}_0$. The results in the figure also suggest that – except in the extreme case when $\alpha = \infty$ – the equilibrium outputs in every market converge to the same limit as $N \to \infty$ independently of the value of α. We formalize this observation in the following proposition under the additional condition $\lim_{N\to\infty} B_1(N) = 0$, which is satisfied for the exponential family above for every $\alpha < \infty$. In other words, we consider an asymptotic regime with a very large number of small firms that collectively have a fixed cumulative budget B_C.

Proposition 2 *Consider the asymptotic regime in which the number of firms N goes to infinity and $\lim_{N\to\infty} B_1(N) = 0$. Then*

$$\lim_{N\to\infty} X_j^*(N) = \beta_j \Big(R_j - H(B_C)\Big)^+ \qquad \textit{for all } j \in [M].$$

It follows that $\lim_{N\to\infty} X_j^*(N) = 0$ for all $j < k^* := \min\{j \geq 1 \colon \mathbb{B}_j < B_C\}$.

It is worth noting that the limiting output quantities in the previous proposition are such that the margin $r_j(X_j^*)$ is constant and equal to $H(B_C)$ for all $j \geq k^*$. Intuitively, in the limit as $N \to \infty$, each firm becomes infinitesimally small and their individual strategies have no impact on a market's return.

We next look at how the value of N and the distribution of budgets across the firms impact the firms' payoffs and consumers' surplus. To this end, we denote by

$$\Pi_{\mathrm{C}} := \sum_{i=1}^{N} \Pi_i = \sum_{j=1}^{M} \left(R_j - \frac{X_j^*}{\beta} \right) X_j^*$$

the firms' cumulative payoff across all markets. Similarly, we define

$$\mathcal{S} := \sum_{j=1}^{M} \frac{(X_j^*)^2}{2\,\beta_j}$$

to be the consumers' total surplus across all markets[5]. Finally, we define the social surplus across all markets to be $\mathcal{W} := \Pi_{\mathrm{C}} + \mathcal{S}$.

Figure 3 depicts the values of Π_{C} (left panel), $\mathcal{S}$ (middle panel) and $\mathcal{W}$ (right panel) as a function of N for four distributions of the total budget B_{C} across firms using (9) and the exponential family with $\alpha \in \{0, 5, 10, \infty\}$. The dashed-line on each panel corresponds to the solution that a social planner would obtain by solving the aggregate social surplus maximization problem:

$$\max_{X_j \geq 0} \sum_{j=1}^{M} \left(R_j - \frac{X_j}{\beta_j} \right) X_j + \sum_{j=1}^{M} \frac{(X_j)^2}{2\,\beta_j} \qquad \text{subject to} \qquad \sum_{j=1}^{M} X_j \leq B_{\mathrm{C}}.$$

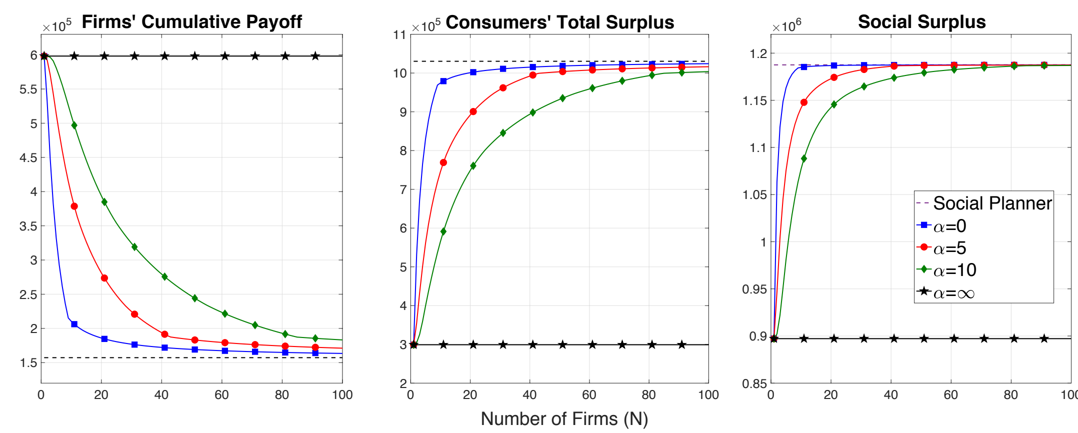


Figure 3: Firms' cumulative payoff (Π_{C}), consumers' total surplus ($\mathcal{S}$) and social surplus ($\mathcal{W}$) as a function of N for four distributions of the total budget B_{C} across firms using (9) and the exponential family with $\alpha \in \{0, 5, 10, \infty\}$. The dashed-line on each panel corresponds to the social planner solution. Each plot corresponds to the average over 100 simulations in which the demand function $r_j(X_j) = R_j - X_j/\beta_j$ on each market $j \in [M]$ was randomly generated with $R_j \sim \mathrm{Uniform}[0, 1000]$ and $\beta_j \sim -\log(\mathrm{Uniform}[0, 1])$. The total budget B_{C} on each simulation was set at $B_{\mathrm{C}} = 0.9\,\mathbb{B}_0$.

As we can see from the figure, and except for the limiting case with $\alpha = \infty$, the firms' total payoff decreases in N while the consumers' surplus increases. In aggregate, the net effect is that the social surplus increases with N. Also, these effects are more pronounced for small value of α, i.e., as the cumulative budget B_{C} is more uniformly distributed across firms.

[5] Recall that the consumers' surplus in a market is the area under the demand curve and above a horizontal line at the equilibrium price.

Remark 1 We note that Theorem 1 and Proposition 1 can be used to do some quick and basic analysis of mergers, a topic of interest to many researchers e.g. Bimpikis et al. (2019). For example, suppose two firms merge and the augmented budget of the merged firm is smaller than the cutoff budget of the j^{th} market. Then it follows from (7) that n_j^* does not change post-merger. It then follows from (6) that the equilibrium ordering quantities in the j^{th} market of all firms which are bigger than the merged firm are unchanged post-merger. □

5 Conclusions and Further Research

In this note we have considered a Cournot competition model where a number of firms compete on quantities across a number of independent product markets. We assume independent linear inverse-demand functions and that each of the firms are budget constrained. Together these assumptions allow us to define an explicit ordering of the firms and markets, and define the notions of augmented and cutoff budgets. We then characterize the unique Cournot equilibrium in terms of these augmented and cutoff budgets.

There are several potential directions for further research. For example, the presence of capacity constraints raises a new set of challenges in terms of how these capacity levels are determined in the first place. Our model assumes that firms' capacities are exogenously given but in practice we might expect that firms will also optimize them. Along the same line, our model leads naturally to the study of market dynamics in which potential new firms can enter the market over time. In this situation, we expect that incumbent firms would use capacity levels to gain market power and deter competition. In addition, it might also be of interest to extend the model to include lower bounds (or fixed costs) on the allocation of capacity among markets. In our current model, firms are able to allocate any amount of capacity on a market. Realistically, however, a firm would only operate in a market if it can ensure a minimum sales volume.

There are several other possible directions. First, can we extend our model to incorporate a network structure that restricts the set of markets in which each firm can participate. This extension may be more tractable than handling a general heterogeneous cost structure which includes the network structure as a special case. Second, can we handle products that are substitutes? We have not been able to solve this problem but it's possible that the ranking of both markets and firms still persists in this case thereby suggesting that some progress may be possible. A third direction is the question of mergers as discussed in Remark 1 of Section 4. A further direction concerns our observation in Section 4 that the consumer surplus and social surplus increased in the number of firms N whilst keeping the total aggregate budget fixed. While we witnessed this behavior numerically for a broad range of budget distributions (as characterized by the parameter α), it would be instructive to establish some of these results more rigorously. We would also like to extend our analysis of the problem with non-linear inverse-demand functions from Appendix C. For example, is it possible to show the iterative algorithm based on Proposition C.1 always converges? Or can we solve explicitly for the equilibrium quantities in some cases? Finally, it would also be of interest to study the stability of the unique Cournot equilibrium and to understand what kind of dynamics or adjustment processes would ensure convergence to the equilibrium.

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A Appendix: Intuition for the Augmented and Cutoff Budget Definitions

In this appendix we provide some intuition regarding the definitions of augmented and cutoff budgets. We begin with the cutoff budgets $\{\mathbb{B}_j\}$. The following result establishes that the cutoff budget $\mathbb{B}_j$ corresponds to the minimum cumulative investment in markets $\{j+1,\ldots,M\}$ that guarantees that market j is open in equilibrium. (By "open" we mean that $X_j > 0$.) This minimum cumulative investment is independent of the number of firms and their budgets.

Proposition 3 *Suppose that in equilibrium $\sum_{k=j+1}^{M} X_k > \mathbb{B}_j$. Then we must have $X_j > 0$.*

Proof of the Proposition: Suppose by contradiction that $\sum_{k=j+1}^{M} X_k > \mathbb{B}_j$ and $X_j = 0$. Then, there exists a market $\ell \in \{j+1,\ldots,M\}$ such that $R_\ell - X_\ell/\beta_\ell < R_j$. Indeed, if such an ℓ does not exist then we would have $R_k - X_k/\beta_k \geq R_j$ for all $k \in \{j+1,\ldots,M\}$ or equivalently $\beta_k [R_k - R_j] \geq X_k$ for all $k \in \{j+1,\ldots,M\}$. Summing over $k \in \{j+1,\ldots,M\}$ we would get $\mathbb{B}_j = \sum_{k=j+1}^{M} \beta_k [R_k - R_j] \geq \sum_{k=j+1}^{M} X_k$ which violates our hypothesis.

Since $R_\ell \geq R_j$ it follows that $X_\ell > 0$ and there exits a firm i such that $x_{i\ell} > 0$ and $x_{ij} = 0$ (since we have assumed $X_j = 0$). Suppose firm i swaps an amount $\epsilon \in (0, x_{i\ell}]$ from market ℓ to market j. Then the net effect on its payoff is

$$\begin{aligned}
\Delta\Pi_i(\epsilon) &= \left[R_\ell - \frac{X_\ell - \epsilon}{\beta_\ell}\right](x_{i\ell} - \epsilon) + \left[R_j - \frac{\epsilon}{\beta_j}\right]\epsilon - \left[R_\ell - \frac{X_\ell}{\beta_\ell}\right] x_{i\ell} \\
&= \left[\frac{x_{i\ell}}{\beta_\ell} + R_j - \left(R_\ell - \frac{X_\ell}{\beta_\ell}\right)\right]\epsilon + o(\epsilon) > 0
\end{aligned}$$

where the strict inequality follows for ϵ sufficiently small. It follows that firm i would like to move some of its budget from market ℓ to market j which contradicts the equilibrium condition. Hence $X_j > 0$ as required. $\square$

Let us now turn to the interpretation of the augmented budgets $\{\mathcal{B}_i\}$. To simplify the exposition, we will assume in what follows that $\beta_j = 1$ for all markets j. We first note the following facts:

Fact 1: Suppose market j is the only market and there are exactly i firms, none of whom are budget constrained. Then the unconstrained Cournot equilibrium quantity invested by each of the i players is $R_j/(i+1)$.

Fact 2: When there are M markets (and still i firms), then in equilibrium each of them will invest up to $(R_k - R_j)/(i+1)$ in the k^{th} market for $k = j+1,\ldots,M$ before investing in market j. That is, they may not sell as much as the unconstrained Cournot equilibrium $R_k/(i+1)$ (from Fact 1) before it becomes profitable to start investing in market j.

Justification: Suppose each of the i firms has already invested $(R_k - R_j)/(i+1)$ in the k^{th} market for $k = j+1,\ldots,M$ and suppose one of these firms is considering investing an additional ϵ in the k^{th} market. Then the payoff in the k^{th} market for this firm will be

$$P_k(\epsilon) := \underbrace{\left(R_k - \left(\frac{i(R_k - R_j)}{i+1} + \epsilon\right)\right)}_{\text{Price in market } k} \underbrace{\left(\frac{(R_k - R_j)}{i+1} + \epsilon\right)}_{\text{Quantity invested by firm}} .$$

In particular, the firm's gain (to order ϵ) from this additional investment of ϵ is given by

$$\begin{aligned} P_k(\epsilon) - P_k(0) &= \left(R_k - \frac{i(R_k - R_j)}{i+1}\right)\epsilon - \frac{(R_k - R_j)}{i+1}\,\epsilon + o(\epsilon) \\ &= R_j\,\epsilon + o(\epsilon) \end{aligned}$$

which is precisely the gain (again to order ϵ) from the first ϵ units invested in market j. The claim follows. □

Now consider the following argument regarding firm i's possible investment in market j. Firm i will begin to invest in market j if and only if its budget B_i can cover what's required for investing in markets $j+1$ to M. But before determining what firm i requires for investing in markets $j+1$ to M we must also consider firms $i+1$ to N and what they invested in those markets. At worst[6] (in terms of firm i's ability to invest in market j), they will have already invested everything in markets $j+1$ to M. So let us assume that firms $i+1$ to N have already invested their entire budgets in markets $j+1$ to M. In particular, let a_k be the total amount invested by firms $i+1$ to N in market k for $k = j+1, \ldots, M$. Then

$$\sum_{k=j+1}^{M} a_k = \sum_{l=i+1}^{N} B_l. \tag{A-1}$$

Adjusting for these smaller firms, we can view the size of market j as being reduced from R_k to $R_k - a_k$ for $k = j+1, \ldots, M$. It follows from Fact 2 above that firm i will invest up to $(R_k - a_k - R_j)/(i+1)$ in the k^{th} market for $k = j+1, \ldots, M$ before investing in market j. As long as B_i covers these investments in markets $j+1$ to M, firm i will invest in market j. That is, firm i will invest in market j if and only

$$\begin{aligned} B_i &\geq \frac{1}{i+1} \sum_{k=j+1}^{M} (R_k - a_k - R_j) \\ \iff B_i &\geq \frac{1}{i+1} \sum_{k=j+1}^{M} (R_k - R_j) - \frac{1}{i+1} \sum_{l=i+1}^{N} B_l \quad \text{by (A-1)} \qquad \text{(A-2)} \\ \iff (i+1)B_i + \sum_{l=i+1}^{N} B_l &\geq \sum_{k=j+1}^{M} (R_k - R_j) \\ \iff \mathcal{B}_i &\geq \mathbb{B}_j. \qquad \text{(A-3)} \end{aligned}$$

Given these arguments, it might make sense to define a new quantity, say a pseudo-budget $B(i,j)$ given by the r.h.s. of (A-2) for each firm i and market j. We could then say that firm i invests in market j if and only firm i's budget B_i is greater than or equal to the pseudo-budget $B(i,j)$. From an economic perspective, this is easier to understand but it means we have $M \times N$ pseudo-budgets. Instead, by bringing terms that only involve firms to the l.h.s. and terms only involving markets to the r.h.s. as in (A-3), we only need to define N terms involving firms (their augmented budgets) and M market terms (their cutoff budgets) for a total of just $M+N$ terms.

[6] The more that firms $i+1$ to N invest in markets $j+1$ to M, the less that firm i will invest in equilibrium in those markets and hence the more he will have available for investing in market j.

B Appendix: Proofs

Proof of Theorem 1: We start by characterizing the best response that firm i uses as a function of the strategies of the other firms. Taking $\mathbf{X}_{i-}$ as fixed, it is straightforward to obtain that the optimal solution to (2) satisfies

$$x_{ij} = \frac{(\beta_j\, R_j - \beta_j\, \lambda_i - X_{ij-})^+}{2}, \tag{B-4}$$

where $\lambda_i \geq 0$ is the Lagrange multiplier corresponding to the i^{th} firm's budget constraint. In particular, $\lambda_i \geq 0$ is the smallest real such that $\sum_{j=1}^{M} x_{ij} \leq B_i$. Given the ordering of the budgets B_i, it follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ when they are chosen optimally. Equation (B-4) and the ordering of the Lagrange multipliers then implies that for each market j, there is a number $n_j \in \{0, 1, \ldots, N\}$ such that $x_{ij} = 0$ for all $i > n_j$. In other words, n_j is the number of firms that operate in market j. We therefore obtain the following system of equations

$$x_{ij} \quad = \quad \beta_j\, R_j - \beta_j\, \lambda_i - X_j, \qquad \text{for} \quad i = 1, \ldots, n_j \tag{B-5}$$

where $X_j = \sum_{i=1}^{n_j} x_{ij}$ is the total budget allocation in market j. For each $j = 1, \ldots, M$, this is a system with n_j linear equations in n_j unknowns which we can easily solve once the n_j's are known. Summing the x_{ij}'s from $i = 1$ to n_j we obtain

$$X_j = \frac{\beta_j}{n_j + 1} \left[n_j\, R_j - \sum_{i=1}^{n_j} \lambda_i \right]. \tag{B-6}$$

Substituting this value of X_j into (B-5), and using the fact that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$, we see the optimal ordering quantities x_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ satisfy

$$x_{ij} = \frac{\beta_j}{n_j + 1} \left[R_j - \lambda_i\, (n_j + 1) + \sum_{k=1}^{n_j} \lambda_k \right]^+. \tag{B-7}$$

To complete the characterization of the Cournot equilibrium we must compute the values of the Lagrange multipliers $\{\lambda_i,\ i = 1, \ldots, N\}$ as well as the n_j's. For reasons that will soon become apparent, it will be convenient to replace the Lagrange multipliers by an equivalent set of unknowns $\{\alpha_i,\ i = 1, \ldots, N\}$ that we define below.

Suppose $x_{ij} = 0$ for some market j. Then (B-4) implies $\beta_j\, R_j - \beta_j\, \lambda_i - X_j \leq 0$ which, after substituting for X_j using (B-6), implies that

$$R_j \leq \lambda_i\, (1 + n_j) - \sum_{k=1}^{n_j} \lambda_k. \tag{B-8}$$

It follows that n_j depends on j only through the value of R_j, that is, $n_j = n_j(R_j)$. If we replace R_j and n_j in (B-8) with a generic $R \in [0, \infty)$ and $n(R)$, respectively, we can define α_i to be that value of R where the i^{th} retailer moves from ordering zero to ordering a positive quantity. It therefore satisfies

$$\alpha_i = \lambda_i\,(1 + n(R)) - \sum_{k=1}^{n(R)} \lambda_k \tag{B-9}$$

and we note the α_i's are non-decreasing in i. Abusing notation slightly, we see[7] that $n(\alpha_i) = i - 1$ and so (B-9) implies

$$\alpha_i = i\,\lambda_i - \sum_{k=1}^{i-1} \lambda_k \quad \text{for} \quad i = 1, \ldots, N. \tag{B-10}$$

Using (B-10) recursively, one can show that

$$\lambda_i = \frac{\alpha_i}{i} + \sum_{k=1}^{i-1} \frac{\alpha_k}{k\,(k+1)}. \tag{B-11}$$

Substituting this expression in (B-7), it follows that for all $i = 1, \ldots, N$

$$\begin{aligned} x_{ij} &= \beta_j \left[\frac{R_j}{1+n_j} - \lambda_i + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j}\right]^+ = \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i} + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} - \sum_{k=1}^{i-1} \frac{\alpha_k}{k\,(k+1)}\right]^+ \\ &= \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i+1} + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} - \sum_{k=1}^{i} \frac{\alpha_k}{k\,(k+1)}\right]^+ \\ &= \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i+1} + \sum_{k=1}^{n_j} \frac{\alpha_k}{k(k+1)} - \sum_{k=1}^{i} \frac{\alpha_k}{k\,(k+1)}\right]^+ \end{aligned} \tag{B-12}$$

where (B-12) follows from the identity

$$\sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} = \frac{1}{1+n_j} \sum_{k=1}^{n_j} \left(\frac{\alpha_k}{k} + \sum_{l=1}^{k-1} \frac{\alpha_l}{l(l+1)}\right) = \sum_{k=1}^{n_j} \frac{\alpha_k}{k(k+1)}$$

which in turns follows from (B-11) and reversing the order of summation. It should be clear from the discussion above that

$$n_j = \max\left\{i \in \{0\} \cup [N] \text{ such that } \alpha_i \le R_j\right\} \tag{B-13}$$

where $\alpha_0 := 0$ and we therefore only need to derive the values of the α_i's to complete the proof of the theorem.

From (B-11) and the ordering of the firms' budgets, it follows that if firm i's budget constraint is not binding then firm k's budget constraint is also not binding, for all $k = 1, \ldots, i$. As a result, if $\sum_{j=1}^{M} x_{ij} < B_i$ then $\lambda_k = 0$ for all $k = 1, \ldots, i$ and (B-11) implies that $\alpha_k = 0$ for all $k = 1, \ldots, i$. Hence, if $\alpha_i > 0$ the budget constraint is binding and $\sum_{j=1}^{M} x_{ij} = B_i$ as required.

To complete the proof, we need to show that α_i is equal to $H(\mathcal{B}_i)$ for all $i \in [N]$. To this end, we

[7] We are assuming that the N budgets are distinct so that $B_{k-1} > B_k$. This then implies $x_i(\alpha_k) > 0$ for all $i \le k-1$. The case where some budgets coincide is straightforward to handle. We also emphasize that the α_i's need not be in the range $[\min_j R_j, \max_j R_j]$.

make use of Lemma 1 stated immediately after this proof. Setting V_j set to 1 in the lemma implies

$$\sum_{j=1}^{M} x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} x_{lj} = \sum_{j\,:\,\alpha_i < R_j} \frac{\beta_j}{i+1} \left(R_j - \alpha_i\right) \tag{B-14}$$

for $i = 1, \ldots, N$. But the budget constraints for the N firms also imply

$$\sum_{j=1}^{M} x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} x_{lj} \leq B_i + \frac{1}{i+1} \sum_{l=i+1}^{N} B_l \tag{B-15}$$

which, when combined with (B-14), leads to

$$\sum_{j\,:\,\alpha_i < R_j} \beta_j \left(R_j - \alpha_i\right) \leq (i+1)\, B_i + \sum_{l=i+1}^{N} B_l = \mathcal{B}_i \tag{B-16}$$

for $i = 1, \ldots, N$. We can use (B-16) sequentially to determine the α_i's. Beginning at $i = N$, we see that the N^{th} firm's budget constraint is equivalent to

$$\sum_{j\,:\,\alpha_N < R_j} \beta_j \left(R_j - \alpha_N\right) \leq \mathcal{B}_N. \tag{B-17}$$

The optimality condition on λ_i implies that it is the smallest non-negative real that satisfies the i^{th} budget constraint. Since the optimal λ_i's are non-decreasing in i, we see from (B-15) that α_i is therefore the smallest real greater than or equal to 0 satisfying the i^{th} budget constraint. Therefore, beginning with $i = N$ we can check if $\alpha_N = 0$ satisfies (B-17) and if it does, then we know the N^{th} budget constraint is not binding. If $\alpha_N = 0$ does not satisfy (B-17) then we set α_N equal to that value (greater than one) that makes (B-17) an equality. In particular, we obtain that the optimal value of α_N is $H(\mathcal{B}_N)$, as desired.

Note that if $\alpha_N = 0$ then none of the budget constraints are binding. In particular, this implies $\alpha_i = 0$ and $\lambda_i = 0$ for all $i = 1, \ldots N$. Moreover, $\alpha_i = H(\mathcal{B}_i)$ must be satisfied for all i since it is true for $i = N$ and since the B_i's are decreasing. Suppose now that the budget constraint is binding for firms $i+1, \ldots, N$ and consider the i^{th} firm. Then the i^{th} firm's budget constraint is equivalent[8] to (B-16) and we can again use precisely the same argument as before to argue that $\alpha_i = H(\mathcal{B}_i)$ holds. $\square$

We now state and prove the lemma that we used for proving Theorem 1.

Lemma 1 *Consider a collection of linear markets* $\boldsymbol{\mathcal{M}} := \{R_j - x/\beta_j \colon j \in [M]\}$ *and the budget allocations* $\{x_{ij}\}$ *in (B-12). Then, for any vector* $(V_1, \ldots, V_M)$ *we have*

$$(i+1) \sum_{j=1}^{M} V_j\, x_{ij} + \sum_{k=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{kj} = \sum_{j\,:\,\alpha_i < R_j} V_j\, \beta_j \left(R_j - \alpha_i\right). \tag{B-18}$$

[8] Equivalence follows because the second terms on either side of the inequality sign in (B-15) are equal by assumption.

Proof of Lemma 1: We first recall that the α_i's are non-decreasing in i. We also note that $n(\alpha_j) = k-1$ for all α_j satisfying $\alpha_{k-1} \leq \alpha_j < \alpha_k$ where j indexes products and k indexes retailers. Setting $\alpha_{N+1} := \infty$ and noting that $x_{ij} = 0$ if $\alpha_i \geq \alpha_j$, we can combine these results and (B-7) to write

$$x_{ij} = \sum_{k=i}^{N} \frac{\beta_j}{k+1} \left[R_j - (k+1)\,\lambda_i + \sum_{j=1}^{k} \lambda_j \right] \mathbb{1}\Big(\alpha_j \in [\alpha_k, \alpha_{k+1})\Big). \tag{B-19}$$

Letting $\Omega_k := \{j : \ \alpha_k \leq \alpha_j < \alpha_{k+1}\}$, we see that (B-19) implies

$$\begin{aligned}
\sum_{l=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{lj} &= \sum_{l=i+1}^{N} \sum_{k=l}^{N} \sum_{j \in \Omega_k} \frac{V_j\, \beta_j}{k+1} \left[R_j - (k+1)\,\lambda_l + \sum_{s=1}^{k} \lambda_s \right] \\
&= \sum_{k=i+1}^{N} \sum_{l=i+1}^{k} \sum_{j \in \Omega_k} \frac{V_j\, \beta_j}{k+1} \left[R_j - (k+1)\,\lambda_l + \sum_{s=1}^{k} \lambda_s \right] \\
&= \sum_{k=i+1}^{N} \sum_{j \in \Omega_k} \frac{V_j\, \beta_j}{k+1} \left[(k-i)\, R_j - (k+1) \sum_{l=i+1}^{k} \lambda_l + (k-i) \sum_{s=1}^{k} \lambda_s \right].
\end{aligned} \tag{B-20}$$

(B-19) also implies

$$\sum_{j=1}^{M} V_j\, x_{ij} = \sum_{k=i}^{N} \sum_{j \in \Omega_k} \frac{V_j\, \beta_j}{k+1} \left[R_j - (k+1)\,\lambda_i + \sum_{l=1}^{k} \lambda_l \right]. \tag{B-21}$$

If we use the convention $\sum_{l=i+1}^{i} \lambda_l = 0$, then the first sum on the right-hand-side of (B-20) can run from $k = i$ to N instead of $k = i+1$ to N. We can then add $1/(i+1)$ times (B-20) with (B-21) to obtain

$$\sum_{j=1}^{M} V_j\, x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{lj} = \sum_{k=i}^{N} \sum_{j \in \Omega_k} \frac{V_j\, \beta_j}{k+1}\, Z_{ikj} \tag{B-22}$$

where

$$\begin{aligned}
Z_{ikj} \;&:=\; \Bigg[R_j - (k+1)\,\lambda_i + \sum_{l=1}^{k} \lambda_l \\
&\qquad\qquad + \frac{(k-i)\, R_j - (k+1)\, \sum_{l=i+1}^{k} \lambda_l + (k-i)\, \sum_{s=1}^{k} \lambda_s}{i+1} \Bigg].
\end{aligned}$$

Some straightforward algebra together with (B-10) can be used to show

$$Z_{ikj} = \frac{k+1}{i+1}\,(R_j - \alpha_i)$$

and so by the definition of Ω_k we can substitute for Z_{ikj} in (B-22) and obtain (B-18). □

Proof of Proposition 1:

(i) Let us prove that $x_{ij}^* > 0$ if and only if $\mathcal{B}_i > \mathbb{B}_j$. First suppose that $\mathcal{B}_i > \mathbb{B}_j$. It follows from (7) that $n_j^* \geq i$ and (6) implies that

$$x_{ij}^* = \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=i+1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \geq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=i+1}^{n_j^*} \frac{H(\mathcal{B}_i)}{k(k+1)} \right]^+$$
$$= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + H(\mathcal{B}_i) \left(\frac{1}{i+1} - \frac{1}{n_j^*+1} \right) \right]^+$$
$$= \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{n_j^*+1} \right]^+ > 0,$$

where the weak inequality uses the facts that the sequence $\{\mathcal{B}_i\}$ is non-increasing in i and the function $H(\cdot)$ is non-increasing and the strict inequality follows from the fact that $R_j = H(\mathbb{B}_j)$ and the assumption $\mathcal{B}_i > \mathbb{B}_j$.

Let us suppose now that $\mathcal{B}_i \leq \mathbb{B}_j$. In this case (7) implies that $n_j^* \leq i-1$ and (6) leads to

$$x_{ij}^* = \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} - \sum_{k=n_j^*+1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \leq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{n_j^*+1})}{i+1} - \sum_{k=n_j^*+1}^{i} \frac{H(\mathcal{B}_{n_j^*+1})}{k(k+1)} \right]^+$$
$$= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{n_j^*+1})}{i+1} - H(\mathcal{B}_{n_j^*+1}) \left(\frac{1}{n_j^*+1} - \frac{1}{i+1} \right) \right]^+$$
$$= \beta_j \left[\frac{R_j - H(\mathcal{B}_{n_j^*+1})}{n_j^*+1} \right]^+ = 0,$$

where the last equality follows $n_j^* + 1 \leq i$ and so $H(\mathcal{B}_{n_j^*+1}) \geq H(\mathcal{B}_i) \geq H(\mathbb{B}_j) = R_j$.

To prove that x_{ij}^* is non-increasing in i note that

$$\begin{aligned}
x_{ij}^* &= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)} \right]^+ \\
&\geq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)} \right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ = x_{i+1j}^*.
\end{aligned}$$

Finally, the fact that x_{ij}^*/β_j is non-decreasing in j follows trivially from (6) and the fact that n_j^* is non-decreasing in j.

(ii) From part (i) we have that $x^*_{ij} > 0$ if and only if $i \le n^*_j$. It follows that

$$
\begin{aligned}
X^*_j &= \sum_{i=1}^{N} x^*_{ij} = \sum_{i=1}^{n^*_j} \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \\
&= \beta_j \, \frac{n^*_j \, R_j}{1+n^*_j} - \beta_j \sum_{i=1}^{n^*_j} \frac{H(\mathcal{B}_i)}{i+1} + \beta_j \sum_{i=1}^{n^*_j - 1} \sum_{k=i+1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} \\
&= \beta_j \, \frac{n^*_j \, R_j}{1+n^*_j} - \beta_j \sum_{i=1}^{n^*_j} \frac{H(\mathcal{B}_i)}{i+1} + \beta_j \sum_{i=1}^{n^*_j} \frac{(i-1)\, H(\mathcal{B}_i)}{i(i+1)} = \beta_j \left[\frac{n^*_j \, R_j}{1+n^*_j} - \sum_{i=1}^{n^*_j} \frac{H(\mathcal{B}_i)}{i(i+1)} \right].
\end{aligned}
$$

(iii) Let us prove that $\sum_{j=1}^{M} x^*_{ij} = B_i$ if and only if $\mathcal{B}_i \le \mathbb{B}_0$ using backward induction over i. To this end, we find convenient to define $m^*_i := \min\{j \in [M] \colon \mathcal{B}_i > \mathbb{B}_j\}$ for $i \in [N]$, so that $x^*_{ij} > 0$ if and only if $j \ge m^*_i$. Also, using the definition of $H(\cdot)$ in Definition 3.2 we have that

$$
H(\mathcal{B}_i) = \frac{\sum_{k=m^*_i}^{M} \beta_k \, R_k - \mathcal{B}_i}{\sum_{k=m^*_i}^{M} \beta_k} \quad \text{for all } i \in [N] \text{ such that } \mathcal{B}_i \le \mathbb{B}_0. \tag{B-23}
$$

•) Suppose $i = N$. We consider two cases: (a) $\mathcal{B}_N \le \mathbb{B}_0$ and (b) $\mathcal{B}_N > \mathbb{B}_0$.

(a) Suppose $\mathcal{B}_N \le \mathbb{B}_0$. In this case, (7) implies that $n^*_j = N$ for all $j \ge m^*_N$ and so (6) implies

$$
\begin{aligned}
\sum_{j=1}^{M} x^*_{Nj} = \sum_{j=m^*_N}^{M} x^*_{Nj} &= \sum_{j=m^*_N}^{M} \beta_j \left[\frac{R_j}{1+N} - \frac{H(\mathcal{B}_N)}{1+N} \right] \\
&= \frac{1}{1+N} \sum_{j=m^*_N}^{M} \beta_j \, R_j - \frac{H(\mathcal{B}_N)}{1+N} \sum_{j=m^*_N}^{M} \beta_j = \frac{\mathcal{B}_N}{1+N} = B_N,
\end{aligned}
$$

where the second-to-last equality uses (B-23) and the last equality follows from the fact that $\mathcal{B}_N = (1+N)\, B_N$ (see equation (4)).

(b) Suppose $\mathcal{B}_N > \mathbb{B}_0$. In this case, $m^*_N = 1$, $n^*_j = N$ for all $j \in [M]$ and $H(\mathcal{B}_N) = 0$. It follows that

$$
\sum_{j=1}^{M} x^*_{Nj} = \sum_{j=1}^{M} \beta_j \left[\frac{R_j}{1+N} - \frac{H(\mathcal{B}_N)}{1+N} \right] = \frac{1}{1+N} \sum_{j=1}^{M} \beta_j \, R_j = \frac{\mathbb{B}_0}{1+N} < \frac{\mathcal{B}_N}{1+N} = B_N.
$$

We conclude that the result holds for $i = N$.

•) Suppose that the result holds for $i+1$, namely, $\sum_{j=1}^{M} x^*_{i+1j} = B_{i+1}$ if and only if $\mathcal{B}_{i+1} \le \mathbb{B}_0$.

•) Let us prove the result for i. We consider again the two cases: (a) $\mathcal{B}_i \le \mathbb{B}_0$ and (b) $\mathcal{B}_i > \mathbb{B}_0$.

(a) Suppose $\mathcal{B}_i \le \mathbb{B}_0$, then $\mathcal{B}_{i+1} \le \mathbb{B}_0$ and by the induction hypothesis $\sum_{j=1}^{M} x^*_{i+1j} = B_{i+1}$. It follows that

$$\sum_{j=1}^{M} x_{ij}^* = \sum_{j=1}^{M} x_{i+1j}^* + \sum_{j=1}^{M} (x_{ij}^* - x_{i+1j}^*) = B_{i+1} + \sum_{j=1}^{M} (x_{ij}^* - x_{i+1j}^*).$$

Now, for all $j \in [M]$ such that $j < m_i^*$, we have that $x_{ij}^* = x_{i+1,j}^* = 0$. On the other hand, for $j \in [M]$ such that $m_i^* \leq j < m_{i+1}^*$, we have $x_{i+1j}^* = 0 < x_{ij}^*$ and $n_j^* = i$. It follows from (6) that

$$x_{ij}^* = \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} \right] = \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{i+1} \right].$$

Finally, for $j \geq m_{i+1}^*$, we have $0 < x_{i+1j}^* \leq x_{ij}^*$ and from (6) we get that

$$\begin{aligned}
x_{ij}^* &= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right] \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} - \frac{H(\mathcal{B}_i)}{i+1} + \frac{H(\mathcal{B}_{i+1})}{i+2} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)} \right] \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1} \right] \\
&= x_{i+1j}^* + \beta_j \left[\frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1} \right].
\end{aligned}$$

As a result

$$\begin{aligned}
\sum_{j=1}^{M} x_{ij}^* &= B_{i+1} + \sum_{j=1}^{M} (x_{ij}^* - x_{i+1j}^*) \\
&= B_{i+1} + \sum_{j=m_i^*}^{m_{i+1}^*-1} \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{i+1} \right] + \sum_{j=m_{i+1}^*}^{N} \beta_j \left[\frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1} \right] \\
&= B_{i+1} + \frac{1}{i+1} \Big[\sum_{j=m_i^*}^{m_{i+1}^*-1} \beta_j R_j + H(\mathcal{B}_{i+1}) \sum_{j=m_{i+1}^*}^{N} \beta_j - H(\mathcal{B}_i) \sum_{j=m_i^*}^{N} \beta_j \Big] \\
&= B_{i+1} + \frac{1}{i+1} [\mathcal{B}_i - \mathcal{B}_{i+1}] = B_i,
\end{aligned}$$

where the second-to-last equality uses (B-23) and the last equality follows from (4).

(b) Suppose $\mathcal{B}_i > \mathbb{B}_0$. In this case, $H(\mathcal{B}_i) = 0$ and $m_i^* = 1$. Using a similar argument as in part (a) one can show that

$$\sum_{j=1}^{M} x_{ij}^* = \sum_{j=m_{i+1}^*}^{M} x_{i+1j}^* + \frac{1}{i+1} \Big[\sum_{j=1}^{m_{i+1}^*-1} \beta_j R_j + H(\mathcal{B}_{i+1}) \sum_{j=m_{i+1}^*}^{N} \beta_j - \sum_{j=1}^{N} \beta_j \Big]. \tag{B-24}$$

Suppose that $\mathcal{B}_{i+1} \leq \mathbb{B}_0$, then (B-24) together with (B-23) and the induction hypothesis imply that

$$\begin{aligned}\sum_{j=1}^{M} x_{ij}^{*} &= B_{i+1} + \frac{1}{i+1}\Big[\sum_{j=1}^{m_{i+1}^{*}-1} \beta_j\, R_j + \sum_{j=m_{i+1}^{*}}^{N} \beta_j\, R_j - \mathcal{B}_{i+1} - \sum_{j=1}^{N} \beta_j\Big]\\ &= B_{i+1} + \frac{1}{i+1}\Big[\sum_{j=1}^{N} \beta_j\, R_j - \mathcal{B}_{i+1}\Big] = B_{i+1} + \frac{1}{i+1}\Big[\mathbb{B}_0 - \mathcal{B}_{i+1}\Big]\\ &< B_{i+1} + \frac{1}{i+1}\Big[\mathcal{B}_i - \mathcal{B}_{i+1}\Big] = B_i.\end{aligned}$$

Suppose now that $\mathcal{B}_{i+1} > \mathbb{B}_0$ then $m_{i+1}^{*} = 1$. It follows from (B-24) and the induction hypothesis that

$$\sum_{j=1}^{M} x_{ij}^{*} = \sum_{j=1}^{M} x_{i+1j}^{*} < B_{i+1}.$$

But since $B_i \geq B_{i+1}$, we conclude that $\sum_{j=1}^{M} x_{ij}^{*} < B_i$. □

Proof of Proposition 2: Recall from Corollary 1 that for fixed N

$$X_j^{*}(N) = \beta_j \left[\frac{n_j^{*}(N)\, R_j}{1 + n_j^{*}(N)} - \sum_{k=1}^{n_j^{*}(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)}\right],$$

where $n_j^{*}(N) = \max\big\{i \in [N] \text{ such that } \mathcal{B}_i(N) > \mathbb{B}_j\big\}$ and $\mathcal{B}_i(N) = i\, B_i(N) + \sum_{k=i}^{N} B_k(N)$. Note that $\mathcal{B}_1(N) = B_1(N) + \sum_{i=1}^{N} B_i(N) = B_1(N) + B_{\mathrm{C}}$ and so $\lim_{N\to\infty} \mathcal{B}_1(N) = B_{\mathrm{C}}$. Thus, we have that $\lim_{N\to\infty} X_j^{*}(N) = 0$ for all $j \in [M]$ such that $\mathbb{B}_j \geq B_{\mathrm{C}}$, i.e., for all $j < k^{*} = \min\{j \geq 1\colon \mathbb{B}_j < B_{\mathrm{C}}\}$.

Suppose $j \geq k^{*}$ and let $\epsilon > 0$ be such that $B_{\mathrm{C}} - \epsilon > \mathbb{B}_{k*}$. We define

$$n_\epsilon(N) := \max\big\{n \in [N] \text{ such that } \mathcal{B}_i(N) \geq B_{\mathrm{C}} - \epsilon \text{ for all } i \leq n\big\}.$$

Since $\mathcal{B}_1(N) = B_1(N) + B_{\mathrm{C}}$ we have that $n_\epsilon(N) \geq 1$ for all N.

We next show that $\lim_{N\to\infty} n_\epsilon(N) = \infty$. We prove this claim by contradiction. Suppose, otherwise, that there exists an increasing integer-valued sequence $\{N_k\}$ and an integer $\bar{n}_\epsilon$ such that $\lim_{k\to\infty} N_k = \infty$ and $\lim_{k\to\infty} n_\epsilon(N_k) = \bar{n}_\epsilon$. It follows that there exists an integer $\bar{k}_\epsilon$ such that $n_\epsilon(N_k) = \bar{n}_\epsilon$ for all $k \geq \bar{k}_\epsilon$ and so $\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) < B_{\mathrm{C}} - \epsilon$ for all $k \geq \bar{k}_\epsilon$. But, for k sufficiently large

$$\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) = (\bar{n}_\epsilon + 1)\, B_{\bar{n}_\epsilon+1}(N_k) + \sum_{i=\bar{n}_\epsilon+1}^{N_k} B_i(N_k) = (\bar{n}_\epsilon + 1)\, B_{\bar{n}_\epsilon+1}(N_k) - \sum_{i=1}^{\bar{n}_\epsilon} B_i(N_k) + \sum_{i=1}^{N_k} B_i(N_k).$$

Taking limit as $k \to \infty$ and noticing that $\bar{n}_\epsilon$ is finite (independent of k) and $B_i(N_k) \downarrow 0$ as N_k goes to infinity, we get

$$\lim_{k\to\infty} \mathcal{B}_{\bar{n}_\epsilon+1}(N_k) = \lim_{k\to\infty} \sum_{i=1}^{N_k} B_i(N_k) = B_{\mathrm{C}} > B_{\mathrm{C}} - \epsilon.$$

But this contradicts $\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) < B_{\rm C} - \epsilon$ for all $k \geq \bar{k}_\epsilon$. We conclude that $\lim_{N\to\infty} n_\epsilon(N) = \infty$. As a direct consequence we get that $\lim_{N\to\infty} n_j^*(N) = \infty$ for all $j \geq k^*$ since $n_j^*(N) \geq n_\epsilon(N)$. Thus, for $j \geq k^*$

$$X_j^*(N) = \beta_j \left[\frac{n_j^*(N)\, R_j}{1 + n_j^*(N)} - \sum_{k=1}^{n_\epsilon(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} - \sum_{k=n_\epsilon(N)+1}^{n_j^*(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} \right].$$

From this expression and the facts that (i) $H(\mathcal{B}_k(N)) \leq H(0) = a_M < \infty$ for all k and N and (ii) $n_\epsilon(N) \to \infty$ and $n_j^*(N) \to \infty$ as $N \to \infty$, we get that

$$\lim_{N\to\infty} X_j^*(N) = w_j \left[R_j - \lim_{N\to\infty} \sum_{k=1}^{n_\epsilon(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} \right].$$

But, $B_{\rm C} - \epsilon \leq \mathcal{B}_k(N) \leq \mathcal{B}_1(N) = B_1(N) + B_{\rm C}$ for all $k \leq n_\epsilon(N)$. It follows that

$$\beta_j \big(R_j - H(B_{\rm C} - \epsilon)\big) \leq \lim_{N\to\infty} X_j^*(N) \leq \beta_j \big(R_j - H(B_{\rm C})\big).$$

From the continuity of H, letting $\epsilon \downarrow 0$ we conclude that

$$\lim_{N\to\infty} X_j^*(N) \leq \beta_j \big(R_j - H(B_{\rm C})\big) \quad \text{for all } j \geq k^*. \;\; \square$$

C Appendix: Nonlinear Inverse-Demand Functions

In this appendix we consider how our analysis of the budget-constrained retailers' game might extend when the inverse demand functions are non-linear. In Appendix C.1 we show by way of a counter-example that in general a unique Cournot equilibrium need not exist when the inverse demand functions are piece-wise linear. Then in Appendix C.2 we consider non-linear inverse-demand functions that are concave and continuously differentiable. We provide a characterization of any Cournot equilibrium in this setting and explain how they might be used to find an equilibrium.

C.1 Multiple Equilibria with Piece-wise Constant Inverse Demand Functions

We begin by showing that it's no longer true in general that a unique Cournot equilibrium exists when we drop the assumption of linear inverse-demand functions. We illustrate this point with a concrete example that considers a simple piecewise linear inverse-demand function.

Example 1 *Consider a problem with two identical firms ($N = 2$) with budgets $B_1 = B_2 = 500$ and one market ($M = 1$) with a piecewise linear demand function given by*

$$r(X) = \begin{cases} 4700 - 8\,X & \text{if } X \leq 500 \\ 1200 - X & \text{if } X \geq 500, \end{cases}$$

where X is the total budget allocated by both firms, that is, $X = x_1 + x_2$ with $0 \leq x_i \leq B_i$ for $i = 1, 2$. We will show that in this situation, there exist two Cournot equilibria: one with $X^* < 500$ and the other one with $X^* > 500$. Towards this end, let us consider two cases:

> **Case (a)**: *Suppose the inverse-demand function is actually linear and equal to $r_a(X) = 4700 - 8\,X$ for all $X \geq 0$. Then, we can apply the result in Corollary 1 in the paper to get a unique Cournot equilibrium in which the budget allocation of the two firms equals*
>
> $$x_1^* = x_2^* = x_a^* = \frac{(1/8)\,(4700 - H(1500))^+}{3}.$$
>
> *Furthermore, $H(1500) = \inf\big\{z \geq 0 \text{ such that } (1/8)\,(4700 - z)^+ \leq 1500\big\} = 0$. It follows that $x_a^* = 195.8\bar{3}$ and the cumulative output is $X_a^* = 2\,x_a^* = 391.\bar{6} < 500$.*
>
> *To show that the output $(x_1^*, x_2^*) = (x_a^*, x_a^*)$ is in fact an equilibrium for the original problem we need to show that it is optimal for a firm to choose x_a^* if the other firm chooses x_a^* when the demand function is given by the piecewise linear function $r(X)$ as above. In other words, that x_a^* satisfies*
>
> $$x_a^* = \operatorname*{argmax}_x r(x_a^* + x)\,x \quad \text{subject to} \quad 0 \leq x \leq B = 500.$$
>
> *Figure 4(a) depicts the value of $r(x_a^* + x)\,x$ in the range $x \in [0, 500]$. As we can see the global maximum of $r(x_a^* + x)\,x$ is achieved when $x = x_a^* = 195.8\bar{3}$.*
>
> **Case (b)**: *Again assume momentarily that the inverse-demand function is linear and equal to $r_b(X) = 1200 - X$ for all $X \geq 0$. Again, we can apply the result in Corollary 1 to get a*

unique Cournot equilibrium in which the budget allocation of the two firms equals

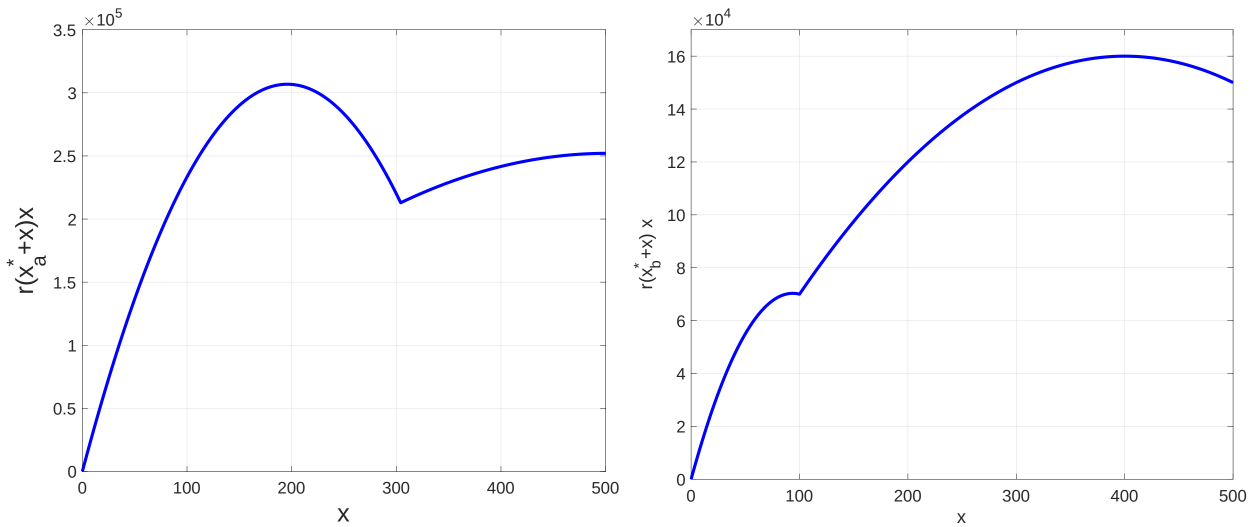
$$x_1^* = x_2^* = x_b^* = \frac{(1200 - H(1500))^+}{3}.$$

In this case, $H(1500) = \inf\left\{z \geq 0 \text{ such that } (1200 - z)^+ \leq 1500\right\} = 0$. *It follows that* $x_b^* = 400$ *and the cumulative output is* $X_b^* = 2\,x_b^* = 800 > 500$.

As in the previous case, to prove that $(x_1^*, x_2^*) = (x_b^*, x_b^*)$ *is an equilibrium for the original problem we need to show that*

$$x_b^* = \operatorname*{argmax}_x r(x_b^* + x)\,x \quad \textit{subject to} \quad 0 \leq x \leq B = 500.$$

Figure 4(b) depicts the value of $r(x_a^* + x)\,x$ *in the range* $x \in [0, 500]$. *We can see that the optimal solution in this case is* $x_b^* = 400$.



(a) (b)

Figure 4: Best response functions for Cases (a) and (b) in Example 1.

We conclude that with the piecewise linear demand function $r(X)$, *there are two equilibria* $(x_1^*, x_2^*) = (x_a^*, x_a^*)$ *and* $(x_1^*, x_2^*) = (x_b^*, x_b^*)$. $\square$

Example 1 shows that if we relax the assumption of linear inverse-demand functions then in general we can no longer guarantee a unique Cournot equilibrium exists. Nonetheless, we can still prove the existence of an equilibrium and partly characterize it under some additional assumptions on the inverse-demand functions. This is the topic of Appendix C.2.

C.2 Characterizing Equilibria with Non-Linear Concave Demand Functions

As in the linear model presented in Section 3, we will use x_{ij} to denote the budget that firm $i \in [N]$ allocates to market $j \in [M]$. Retailer i's profit in market j is equal to $r_j(X_j)\,x_{ij}$, where

$X_j := \sum_{i=1}^N x_{ij}$ is the total budget allocated to market j by all N retailers. We also continue to assume the retailers' budgets are ordered so that $B_1 \geq B_2 \geq \cdots \geq B_N > 0$. We will make the following assumption.

Assumption C.1 *For all $j \in [M]$ there exists $\overline{X}_j > 0$ such that the market return $r_j(x)$ is a continuously differentiable non-negative strictly decreasing and concave function in $[0, \overline{X}_j)$ and $r_j(x) \leq 0$ for $x \geq \overline{X}_j > 0$.*

We define $\mathcal{X} := [0, \overline{X}_1] \times \cdots \times [0, \overline{X}_N]$ to be the set of admissible aggregate budget allocations. Recall that $X_{ij-} = X_j - x_{ij}$ denotes the total budget allocated to market j by all retailers except for retailer i and define $\mathbf{X}_{i-} := (X_{i1-} \ \ldots \ X_{iM-})$. Also, for a given value $\mathbf{X}_{i-}$, retailer i's best-response budget-allocation strategy $\{x^*_{ij}(\mathbf{X}_{i-})\}_{j\in[M]}$ solves the optimization problem:

$$\Pi_i(\mathbf{X}_{i-}) := \max_{x_{ij}\geq 0} \sum_{j=1}^{M} r_j(X_{ij-} + x_{ij})\, x_{ij} \qquad \text{subject to} \qquad \sum_{j=1}^{M} x_{ij} \leq B_i. \tag{C-1}$$

For the sake of completeness, we also recall our definition of a Cournot equilibrium.

Definition C.1 *Consider a collection of markets $\boldsymbol{\mathcal{M}} := \{r_j(\cdot)\colon j \in [M]\}$ and a set of N retailers with budgets $\mathbf{B} = (B_1, \ldots, B_N)$. A* Cournot equilibrium *is a set of budget allocations $\{x^*_{ij}\}_{j\in[M]}$ for $i = 1, \ldots, N$ chosen by the retailers so that:*

(i) *$\{x^*_{ij}\}_{j\in[M]}$ solves the optimization problem (C-1) for $i = 1, \ldots, N$.*

(ii) *They satisfy the fixed-point condition:*

$$X_{ij-} = \sum_{k\neq i} x^*_{kj}(\mathbf{X}_{k-}) \qquad \textit{for all } i = 1, \ldots, N \quad \textit{and} \quad j = 1, \ldots, M. \tag{C-2}$$

We have the following proposition which, in addition to guaranteeing the existence of an equilibrium, characterizes any Cournot equilibrium.

Proposition C.1 *Consider a collection of markets $\boldsymbol{\mathcal{M}} := \{r_j(\cdot)\colon j \in [M]\}$ satisfying the conditions in Assumption C.1. Suppose $\{X_j\}_{j\in[M]}$ and $\{\hat{\lambda}_i\}_{i\in[N]}$ satisfy*

$$X_j\, r'_j(X_j) + \sum_{i=1}^{N} \big(r_j(X_j) - \hat{\lambda}^+_i\big)^+ = 0 \quad j \in [M] \tag{C-3}$$

and

$$B_i + \sum_{j=1}^{M} \frac{(r_j(X_j) - \hat{\lambda}_i)^+}{r'_j(X_j)} = 0 \quad i \in [N]. \tag{C-4}$$

Then, the budget allocations

$$x_{ij} = -\frac{\big(r_j(X_j) - \hat{\lambda}^+_i\big)^+}{r'_j(X_j)}, \qquad i \in [N],\ j \in [M] \tag{C-5}$$

constitute a Cournot equilibrium. Furthermore, there always exists a Cournot equilibrium.

Proof of Proposition C.1: Under Assumption C.1 retailer's i best response problem in (C-1) is a concave optimization problem. As a result, we can we use the following first-order KKT conditions to characterize an optimal solution.

$$
\begin{aligned}
& x_{ij}\, r_j'(X_j) + r_j(X_j) + \beta_{ij} - \lambda_i = 0 && \text{(first-order optimality condition)} \\
& \sum_{j=1}^{M} x_{ij} \le B_i, \quad x_{ij} \ge 0 && \text{(primal feasibility)} \\
& \lambda_i \ge 0, \quad \beta_{ij} \ge 0 && \text{(dual feasibility)} \\
& \lambda_i \left(B_i - \sum_{j=1}^{M} x_{ij} \right) = 0, \quad \beta_{ij} x_{ij} = 0, && \text{(complementary slackness)}
\end{aligned}
\tag{KKT}
$$

where λ_i and β_{ij} are the Lagrange multipliers of retailer's i budget constraint and non-negative constraint $x_{ij} \ge 0$, respectively.

Combining the first-order optimality condition together with the second complementary slackness condition $\beta_{ij} x_{ij} = 0$, the non-negativity of x_{ij} and the fact that $r_j'(X_j) < 0$ (by Assumption C.1) we see that

$$x_{ij} = \frac{(r_j(X_j) - \lambda_i)^+}{-r_j'(X_j)}.$$

Summing over $i \in [N]$ we get that at optimality

$$X_j\, r_j'(X_j) + \sum_{i=1}^{N} \big(r_j(X_j) - \lambda_i\big)^+ = 0 \quad j \in [M]. \tag{C-6}$$

Let use define $\hat{\lambda}_i$ to be unique the solution of

$$B_i + \sum_{j=1}^{M} \frac{(r_j(X_j) - \hat{\lambda}_i)^+}{r_j'(X_j)} = 0, \tag{C-7}$$

Then, combining the value of x_{ij} above and the first complementary slackness condition we obtain

$$\lambda_i \left(B_i + \sum_{j=1}^{M} \frac{(r_j(X_j) - \lambda_i)^+}{r_j'(X_j)} \right) = 0 \quad \Longrightarrow \quad \lambda_i = \hat{\lambda}_i^+.$$

It follows that we can rewrite (C-6) as

$$X_j\, r_j'(X_j) + \sum_{i=1}^{N} \big(r_j(X_j) - \hat{\lambda}_i^+\big)^+ = 0 \quad j \in [M]. \tag{C-8}$$

In sum, if $\{X_j\}_{j\in[M]}$ and $\{\hat{\lambda}_i\}_{i\in[N]}$ jointly solve (C-7)-(C-8) then

$$x_{ij} = \frac{(r_j(X_j) - \hat{\lambda}_i^+)^+}{-r_j'(X_j)}, \quad \lambda_i = \hat{\lambda}_i^+ \quad \text{and} \quad \beta_{ij} = \lambda_i - r_j(X_j) - x_{ij}, r_j'(X_j) \qquad i \in [N],\ j \in [M]$$

solve the first-order KKT conditions.

To prove the existence of $\{X_j\}_{j\in[M]}$ and $\{\hat{\lambda}_i\}_{i\in[N]}$ that solve (C-7)-(C-8), we invoke Brouwer's fixed-point theorem. To this end, we define (implicitly) the following continuous mappings:

- $F : \mathcal{X} \to \mathbb{R}^N$ such that for $X \in \mathcal{X}$ the vector $\hat{\lambda} = F(X)$ solves (C-7). The fact that F is a well-defined continuous mapping follows from the fact that for a given X_j the left-hand side in (C-7) is continuously increasing from from $-\infty$ to $B_i > 0$ as $\hat{\lambda}_i$ increasing from $-\infty$ to $\max_j\{r_j(X_j)\}$.
- $G : \mathbb{R}^N \to \mathcal{X}$ such that for $\hat{\lambda} \in \mathbb{R}^N$ the vector $X = G(\hat{\lambda})$ solves (C-8). To see that the mapping G is also well-defined note that the left-hand side of (C-8) is monotonically decreasing from $\sum_{i\in[N]}(r_j(0) - \hat{\lambda}_i^+)^+$ to $\bar{X}_j\, r_j'(\bar{X}_j) < 0$ as X_j increases from 0 to $\bar{X}_j$.

As a result, $G \circ F : \mathcal{X} \to \mathcal{X}$ is a continuous mapping from the convex compact set $\mathcal{X}$ into itself and by Brouwer's fixed-point theorem there exists a fixed point $X^* \in \mathcal{X}$ such that $X^* = G \circ F(X^*)$. It follows that X^* and $\hat{\lambda}^* = F(X^*)$ solves (C-7)-(C-8) and therefore define a Cournot equilibrium. $\square$

Recall that we have ranked the value of the retailers budgets so that $B_1 \geq B_2 \geq \cdots \geq B_N$. The following corollary follows directly from (C-7):

Corollary C.1 *The values $\{\hat{\lambda}_i\}_{i\in[N]}$ in Proposition C.1 satisfy $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_N$. As a result, the budget allocations in a Cournot equilibrium satisfy $x_{1j} \geq x_{2j} \cdots \geq x_{Nj}$ for all $j \in [M]$.*

C.2.1 An Iterative Scheme to Find a Cournot Equilibrium

Note that it is very easy to iterate on (C-3) and (C-4) to try and find a fixed point, i.e. an equilibrium. For a fixed aggregate allocation X_j, then solving for the $\hat{\lambda}_i$'s via (C-4) requires numerically solving N separate 1-dimensional equations. Moreover, for each i, it is easy to check if $\hat{\lambda}_i < 0$ (in which case we can find a closed-form expression for it), and if it's not then we can do a simple binary search for $\hat{\lambda}_i \in [0, \max_j r_j(X_j)]$. Once the $\hat{\lambda}_i$'s have been found we can then solve (C-3) for the X_j's. This is easy to do since (C-3) will decouple into M separate 1-dimensional equations. Again, it is easy to see that a binary search on $X_j \in [0, \overline{X}_j]$ will suffice to solve for each X_j. We could initialize the search easily, for example by setting $\hat{\lambda}_i = 0$ for all i or setting $X_j = 0$ for all j . Alternatively, we could set the X_j's according to a central planner's solution.

While we cannot guarantee that this iterative scheme will converge, in unreported numerical experiments we found it to converge very rapidly to the unique Cournot equilibrium when we assumed linear inverse-demand functions.