

**IEOR E4602: Quantitative Risk Management (Spring 2016)**  
Columbia University  
**Instructor: Martin Haugh**  
**Assignment 3: Due Monday 29<sup>th</sup> February 2016**

**Answer Questions 5, 6 and three of the remaining four questions.**

Questions 1 and 2 are taken from *MFE* but consulting the textbook will not help: the only details provided there are those given in the questions below.

**Question 1 (Example 5.22 in 1st ed. of *MFE*)**

Let  $X_1$  be a positive-valued random variable and let  $X_2 := 1/X_1$  and  $X_3 := \exp(-X_1)$ . Clearly  $(X_1, X_2)$  and  $(X_1, X_3)$  are countermonotonic random vectors while  $(X_2, X_3)$  is comonotonic. Let  $C$  be the copula of  $(X_1, X_2, X_3)$ .

- (a) Explain why  $C$  is also the copula of  $(X_1, -X_1, -X_1)$ .
- (b) Now show that  $C(u_1, u_2, u_3) = \max\{\min\{u_2, u_3\} + u_1 - 1, 0\}$ .

**Question 2 (Example 5.26 in 1st ed. of *MFE*)**

Let  $X_1$  and  $X_2$  be two random variables such that  $\ln(X_1) \sim N(0, 1)$  and  $\ln(X_2) \sim N(0, \sigma^2)$  where  $\sigma > 0$ .

- (a) Explain why the log-normally-distributed random variables,  $X_1$  and  $X_2$ , are not of the same type when  $\sigma \neq 1$ .
- (b) Let  $\rho_{min}$  and  $\rho_{max}$  be the minimum and maximum possible correlations, respectively, of  $X_1$  and  $X_2$ . Explain why it must be the case that  $\rho_{min} > -1$  and  $\rho_{max} < 1$ .
- (c) Let  $Z \sim N(0, 1)$  and show that if  $(X_1, X_2) = (e^Z, e^{\sigma Z})$  then  $X_1$  and  $X_2$  are comonotonic. Similarly show that if  $(X_1, X_2) = (e^Z, e^{-\sigma Z})$  then  $X_1$  and  $X_2$  are countermonotonic.
- (d) Now calculate  $\rho_{min}$  and  $\rho_{max}$  as a function of  $\sigma$ . Plot both of these quantities as a function of  $\sigma$  in the interval  $\sigma \in [.1, 5]$ . What do you conclude?

**Question 3**

Write a function to plot the density of (i) the bivariate Gaussian copula and (ii) the bivariate  $t$ -copula for  $\nu = 4$  degrees-of-freedom. Your function should take the correlation,  $\rho$ , as an input. Run your function for different values of  $\rho$ . What do you notice?

**Question 4**

- (a) Write a function that simulates  $n$  samples of a bivariate  $t$  distribution with  $\nu$  degrees-of-freedom, mean vector  $\mu$  and correlation  $\rho$  and then calculate estimates of  $\rho$  using (i) the usual Pearson correlation coefficient and (ii) Kendall's  $\tau$ .
- (b) Write another function that calls your function from part (a) 2000 times with  $n = 60$ ,  $\rho = .5$  and  $\mu = \mathbf{0}$ , and then create two plots corresponding to the estimates produced by

methods (i) and (ii) respectively. What estimator do you prefer?

### Question 5

Let  $S_1, \dots, S_N$  denote the time  $T$  prices of  $N$  securities all of which have been normalized so that their time  $t = 0$  price is 1. Suppose we wish to price a particular basket option that expires at time  $T$  with a payoff of

$$\max(0, S_{(i)} - K) \tag{1}$$

where  $S_{(i)}$  is the  $i^{\text{th}}$  largest security price at time  $T$ . For example if we take  $i = 1$  then the option is a *call-on-the-max* option whereas if  $i = 2$ , then the option is a *call-on-the-second-max*. Assume that the risk-neutral distribution of each  $S_j$  satisfies

$$\ln(S_j) \sim N\left((r - \sigma_j^2/2)T, \sigma_j^2 T\right) \tag{2}$$

where  $r$  is the risk-free interest rate. Note that (2) implies each stock is non-dividend paying and follows a geometric-Brownian motion under the risk-neutral probability measure. (In practice, we would not use a log-normal distribution for the  $S_j$ 's but as we want to focus on the dependency structure, that assumption is fine here.)

**(a)** Write a Monte-Carlo simulation to price the basket option with  $T = 1$  year,  $i = 1$ ,  $r = 2\%$ ,  $N = 10$  and  $\sigma_1 = \dots = \sigma_N = .3$ . You should assume that the risk-neutral dependency structure is given by a Gaussian copula with an equicorrelation matrix with  $\rho = .5$ . Your code should output the option price for values of  $K = 1.6, 1.65, 1.7, \dots, 2$ . (In order to make your Monte-Carlo efficient you might consider using variance reduction techniques such as importance sampling.)

**(b)** Repeat part (a) but now assume that the dependency structure is given by a multivariate  $t$  copula with 4 degrees-of-freedom and the same equicorrelation matrix.

**(c)** Repeat parts (a) and (b) but now set  $i = 2$ . What, if anything do you notice? (You should ensure that your statistical error is sufficiently small as to render comparisons between the prices under the two dependence structures possible.)

### Question 6

This question is based on the methodology developed by *CreditMetrics* for evaluating the credit risk of corporate bond portfolios and other credit sensitive securities. The goal in this question is to estimate the 1% VaR of a portfolio of risky bonds. All of the data for this question may be downloaded from the Excel spreadsheet *CorporateBondPortfolioData.xls* that has been posted on *CourseWorks*. Moreover, a Matlab file, *CorpBondPortfolio.m*, has also been posted on *CourseWorks*. This file reads all of the relevant data in from the spreadsheet and contains some comments and commands that you can use to complete the task.

The portfolio consists of 20 corporate bonds of varying maturities, face values, coupons, maturities, and *credit ratings*. In any given year, the rating of a corporate bond may change from one level to another. The dynamics of this process are given by the *Ratings Transition Matrix* in the spreadsheet. Note that we are implicitly assuming that ratings dynamics are governed by a Markov chain where the current rating is sufficient to determine the probability distribution of next year's rating. The spreadsheet also contains the 1-year *forward curves* that should be used to value bonds of differing ratings one year from now. Finally the spreadsheet contains a  $20 \times 20$  correlation matrix. This correlation matrix is the correlation matrix of the *Gaussian copula* that is used to determine the *joint* transition ratings of the 20 bonds. (The marginal ratings are given by the *Ratings Transition Matrix*. If a bond defaults over the next year you may assume that a recovery rate of  $R = 40\%$  applies so that the investor will be left with a bond that is worth 40% of the bond's face value. (We could easily allow for a random recovery value that varies with the bond's *seniority class* but we will just take it to be constant here.)

(a) Simulate  $n = 10,000$  scenarios for the portfolio gain / loss one year from now. This gain / loss should be calculated **relative to the value of the portfolio one year from now if none of the ratings changed**. Estimate the mean, standard deviation and 99% VaR and CVaR for this portfolio. Compute approximate 95% confidence intervals for the mean and standard deviation. (An approximate confidence interval for the 99% VaR and CVaR could also be calculated. It is more complicated, however, so you are not expected to do this.)

(b) If you assumed that the portfolio gain / loss is a given by a normal distribution with the same mean and standard deviation as you estimated in part (a), what would your estimate be for the 99% VaR?

**Remarks:** Note that the model we have developed in this question can easily be used to simulate the actuarial values of CDO tranches or indeed, portfolios of CDO tranches, possibly on different reference portfolios. We could do this by simulating the ratings transitions of the bond portfolio over multiple periods instead of just a single period. Moreover, we could easily have substituted the  $t$  or other copulas in place of the Gaussian copula that we used. It can easily be generalized in other directions as well. As such, the modeling technology developed here has many applications and can be of considerable value. Finally, it is worth

mentioning that while the ratings agencies have been justifiably criticized in recent times for their failure to accurately rate credit instruments, most of these criticisms apply to their ratings of asset-backed securities and not to corporate bonds. At the same time, there have been some spectacular ratings failures even with corporate bonds. *MBIA* is a particularly egregious example that played a very significant role in the 2008 credit crisis! As a result, if you wish to employ the ratings transitions technology developed in this question, you need to be alert and correct any ratings that are clearly ridiculous and known by the market-place to be so.