IEOR E4602: Quantitative Risk Management (Spring 2016)

Columbia University

Instructor: Martin Haugh

Assignment 2: Due Wednesday 17^{th} February 2016

Question 1 (Example 6.32 from *MFE, 2nd ed.*)

Let $\mathbf{X}$ be an n-dimensional random vector with standardized margins (zero mean and unit variance) and an *equicorrelation* matrix, $\mathbf{\Sigma}$, where all off-diagonal elements are equal to a constant, $\rho > 0$. This means that $\mathbf{\Sigma}$ may be written as $\mathbf{\Sigma} \;=\; \rho \mathbf{J}_n \;+\; (1-\rho)\mathbf{I}_n$ where $\mathbf{J}_n$ is an $n \times n$ matrix of 1's and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Note that $\mathbf{\Sigma}$ has the form $\mathbf{\Sigma} \;=\; \mathbf{B}\,\mathbf{\Omega}\,\mathbf{B}^T \;+\; \mathbf{\Upsilon}$, as discussed in the lecture notes with $\mathbf{B} = \sqrt{\rho}\,\mathbf{1}$ where $\mathbf{1}$ is a vectors of 1's.

Show that we obtain a 1-factor decomposition, $\mathbf{X} = \mathbf{B}F \;+\; \epsilon$, if we set

$$
\begin{aligned}
F &= \frac{\sqrt{\rho}}{1+\rho(n-1)} \sum_{i=1}^{n} X_i \;+\; \sqrt{\frac{1-\rho}{1+\rho(n-1)}}\,Y \quad \text{and} \\
\epsilon_i &= X_i \;-\; \sqrt{\rho}\,F \qquad \text{for} \;\; i = 1, \ldots, n
\end{aligned}
$$

where Y is any zero-mean, unit-variance random variable that is independent of $\mathbf{X}$.

Question 2

Let $\mathbf{A}$ be an $n \times n$ equicorrelation matrix with all off-diagonal elements equal to a constant, ρ. We know such a matrix contains n independent eigen vectors.

(a) Show that $\mathbf{e}_1 = (1, \ldots, 1)^T$ and $\mathbf{e}_i = (1, 0, \ldots, 0, -1, 0, \ldots, 0)^T$ for $i = 2, \ldots, n$ constitute a system of n such independent eigen vectors. (Note that the i^{th} component of $\mathbf{e}_i$ equals -1 and every other component is zero except for the first which is equal to 1.)

(b) What are the eigen-values of $\mathbf{A}$? How would you interpret these eigen values for moderate to large values of ρ? What is the minimum value of ρ for which $\mathbf{A}$ remains a correlation matrix?

(c) In practice one often wishes to perturb or *bump* a given correlation matrix, $\mathbf{R}$ say. ($\mathbf{R}$ might be required to price a basket option or some credit derivative for example.) It is important to ensure that the perturbed correlation matrix indeed remains a correlation matrix. There are several ways to perform such a stress. One possibility is as follows:

(1) Compute the spectral decomposition of $\mathbf{R}$ so that we obtain $\mathbf{R} \;=\; \mathbf{\Gamma}\,\mathbf{\Delta}\,\mathbf{\Gamma}^T$ where $\mathbf{\Gamma}$ is the matrix of eigen vectors and $\mathbf{\Delta}$ is the corresponding diagonal matrix of eigen-values, ordered so that $\lambda_1 \geq \ldots \geq \lambda_n$.

(2) Set $\mathbf{\Delta} = \mathbf{\Delta}^*$ where $\mathbf{\Delta}^*$ is the same as $\mathbf{\Delta}$ except that λ_1 is replaced by $\lambda_1 + \epsilon$, for some real ϵ.

(3) Set $\mathbf{R}^* := \mathbf{\Gamma}\,\mathbf{\Delta}^*\,\mathbf{\Gamma}^T$. Note that $\mathbf{R}^*$ will not be a correlation matrix. Why?

(4) Set $\bar{\mathbf{R}} := \mathbf{D}^{-1}\, \mathbf{R}^*\, \mathbf{D}^{-1}$ where $\mathbf{D}$ is a diagonal matrix with $D_{i,i} = \sqrt{\mathbf{R}^*{}_{i,i}}$.

Investigate the success of this technique using the Dow Jones data that is available in the *fBasics* package in `R`. The data can be accessed as follows in `R`:

```
> data("DowJones30")
> x = DowJones30[,2:31]
```

(Note that the data contains approximately 10 years worth of data so feel free to use a shorter time window if you prefer.) As part of your investigation, you should check if a positive value of ϵ results in an increase in the average correlation. Does it increase every pairwise correlation? What happens if ϵ is negative? In the light of part (b) are you surprised by this?

Note that this technique may be used more generally for creating correlation matrices from matrices that look like correlation matrices but are not, for example, positive semi-definite. There are other methods for stressing correlation matrices that we may introduce in a later assignment.

Question 3

Download the spreadsheet *USRiskFreeRatesWeekly.xls* from CourseWorks. The spreadsheet consists of weekly default-free spot interest rates for the following maturities: 1, 3 and 6 months, and 1, 2, 3, 5, 7, 10 and 20 years. The data is available between August 2001 and February 2010. This question can be done in any of `Matlab`, `R` etc.

(a) Write a piece of code that performs a Principal Components Analysis on the interest rate data in the *USRiskFreeRatesWeekly.xls* spreadsheet. Your code should be general enough to perform the PCA on the data in a given time window. In addition to the factor loadings, your code should (i) check that the spectral decomposition is correct, i.e. that $\mathbf{\Sigma} = \mathbf{\Gamma}\, \mathbf{\Delta}\, \mathbf{\Gamma}^T$, and (ii) compute the cumulative percentage of the variability that is explained by the principal components. (Remember to perform your analysis on the *changes* in interest rates.)

(b) Run your code for the following time windows: 2007, 2008 and 2009. What do you notice?

Question 4

Suppose I have a portfolio consisting of $\$1m$ notional, i.e. face-value, of each of 1-year, 10-year and 20-year zero-coupon default-free bonds.

(a) Compute the current value of my portfolio using the spot rate curve as of January 29th 2010 in the *USRiskFreeRatesWeekly.xls* spreadsheet.

(b) Use the results of your PCA analysis in the previous question for 2009 to estimate the 95% and 99% VaR and CVaR of your portfolio. You should use just the first three principal components and assume that they have a multi-variate normal distribution with a diagonal variance-covariance matrix whose entries are given by the the first three eigen-values.

(c) Use a historical Monte-Carlo to estimate the 95% and 99% VaR and CVaR of your portfolio using the rate data from all three years. Compare your answers with part (b).