IEOR E4703 Monte-Carlo Simulation
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Due: 11.00am Monday 6 March 2017

Assignment 4

1. Consider the cumulant generating function, $\psi(t)$, as defined in the paragraph immediately following Example 9 of the *Further Variance Reduction Methods* lecture notes. We will adopt the same setting as described in that paragraph so that $S_n = \sum_{i=1}^n X_i$ where the X_i's are IID with $\mathbb{E}[X_1] < 0$. We wish to estimate $\theta := P(\tau_a < \infty)$ where $\tau_a := \min\{n \geq 0 : S_n \geq a\}$. Then the expression in the lecture notes states that

$$\theta = \mathbb{E}_t \left[I_{\{\tau_a < \infty\}} e^{-tS_{\tau_a} + \tau_a \psi(t)} \right]$$

where $\psi(t) := \log(M_X(t))$ is the cumulant generating function and $M_X(t) := \mathbb{E}[e^{tX}]$ is the moment generating function of X. We also assume that $\psi(t)$ is finite in a neighborhood of the origin.

   (a) Show that $\psi'(t) = \mathbb{E}_t[X]$, where X is a generic random variable with the same distribution as the X_i's.

   (b) Show that $\psi''(t) = \mathrm{Var}_t(X)$ so that $\psi(t)$ is convex.

   (c) Suppose there exists some t such that $0 < \psi'(t) < \infty$. Explain why this implies $t > 0$ and why we can therefore write $\theta = \mathbb{E}_t \left[e^{-tS_{\tau_a} + \tau_a \psi(t)} \right]$. (Drawing a graph of $\psi(t)$ helps.)

   (d) Suppose there exists a $t > 0$ such that $\psi(t) > 0$. Explain why there must be a unique $t^* > 0$ such that $\psi(t^*) = 0$. (Again a graph of $\psi(t)$ helps.)

   (e) Show that $\theta = \mathbb{E}_{t^*} \left[e^{-t^* S_{\tau_a}} \right]$ which in turn implies $\theta \leq e^{-t^* a}$. (In fact it can also be shown that $\theta = O(e^{-t^* a})$.)

   (f) It is clear that the second moment of the estimator, $e^{-t^* S_{\tau_a}}$, sampled using the tilted density f_{t^*}, satisfies $\mathbb{E}_{t^*} \left[e^{-2t^* S_{\tau_a}} \right] \leq e^{-2t^* a}$. Argue that importance sampling using t^* is asymptotically optimal as $a \to \infty$. (This is just a one-line argument based on the fact that $\mathbb{E}[Z^2] \geq \mathbb{E}[Z]^2$ for all random variables, Z.)

2. Consider an insurance company that receives total premia of c dollars per unit time and incurs losses that arrive according to a Poisson process with arrival rate, λ. Assume that loss sizes are IID with CDF, F, and that they are independent of the Poisson arrival process. Assume also that the insurance company has a capital reserve of A dollars. Let θ be the probability that the insurance company ever goes bankrupt. Describe a detailed importance sampling algorithm for estimating θ when the loss sizes have an exponential distribution with parameter, δ. (The results of Question 1 should guide your choice of importance sampling density.)

3. Implement your algorithm from Question 2 and estimate θ when $\lambda = 5$, $\delta = 1/100$ (so the average claim size is \$100), $c = \$600$ and $A = \$10,000$. Use $N = 100,000$ paths and also compute an approximate 95% confidence interval for θ.

4. Give a stratification procedure to estimate the integral

$$\theta := \int_0^1 e^{x^2}\,dx.$$

Use 10 equiprobable strata for whichever stratification random variable you choose and compare the estimated variance of the stratification estimator with that of the standard estimator.

5. Consider the problem of pricing an Asian call option on a stock with time t price given by S_t. You may assume that $S_t \sim GBM(r, \sigma)$ where $r = .05$ and $\sigma = .25$. Other relevant parameters are $T = 1$ year, $S_0 = 100$ and $m = 6$ averaging periods. We assume $K = 90$ so the option is in-the-money. We will use a stratification method which is useful for in-the-money options. It is convenient to write the payoff function as

$$h(Z_1, \ldots, Z_m) = \max\left(0, g(Z_1, \ldots, Z_m) - K\right)$$

where

$$\begin{aligned} g(Z_1, \ldots, Z_m) &= \frac{\sum_{i=1}^m S_{\frac{iT}{m}}}{m} \\ &= \frac{\sum_{i=1}^m S_0 e^{(r-\sigma^2/2)iT/m + \sigma\sqrt{T/m}(\sum_{j=1}^i Z_j)}}{m} \end{aligned}$$

where the Z_j's are IID $\mathrm{N}(0,1)$.

(a) Let $\mathbf{grad}(\mathbf{0})$ be a vector such that the i^{th} component of this vector contains the partial derivative of $g(z_1, \ldots, z_m)$ with respect to z_i, evaluated at $\mathbf{0} = (0, \ldots, 0)$. Let $a = e^{(r-\sigma^2/2)T/m}$ and let $b = \sigma\sqrt{T/m}$. Show that the i^{th} component of $\mathbf{grad}(\mathbf{0})$ is given by

$$[\mathbf{grad}(\mathbf{0})]_i = \frac{bS_0(\sum_{j=i}^m a^j)}{m}.$$

(b) Let $\mu = (\mu_1, \ldots, \mu_m)$ be given by

$$\mu = \frac{\mathbf{grad}(\mathbf{0})}{||\mathbf{grad}(\mathbf{0})||}$$

where $||.||$ is the usual distance norm. Also, let $W = \sum_{i=1}^m \mu_i Z_i$. Show that W may be used as a stratification variable.

(c) Write a program to implement a stratification method using the stratification variable W. Use 10 equiprobable strata and 1000 samples for each stratum to estimate the option price and build 99% confidence intervals using the stratification method.

(d) How many samples will be needed by the usual simulation method to achieve the same accuracy as the stratification method of part (c)? To answer this part, you will need to write and run a regular Monte-Carlo program for pricing the option.