IEOR E4703 Monte-Carlo Simulation
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Due: 12.55pm Thursday 23 February 2017

Assignment 3

1. Let X and Y be independent exponentials with X having mean 1 and Y having mean 2, and suppose we want to use simulation to estimate $\theta := \mathrm{P}(X + Y \geq 4)$. If you were going to use conditional expectation to reduce the variance of the estimator, would you condition on X or Y? Explain your reasoning.

2. Consider Example 10 of the *Simulation Efficiency and an Introduction to Variance Reduction Methods* lecture notes. Let $S_0 = 100$, $T = 1$ year, $r = 0.05$, $\sigma = .25$, $L = 105$, $K_1 = 110$ and $K_2 = 120$.

   (a) Using 10,000 replications, estimate the price of the option using the "usual" simulation algorithm. Compute an approximate 95% confidence interval for the option price.

   (b) Now use the method of conditional expectation and 10,000 replications to estimate the option price again. (You will need to use the Black-Scholes option pricing formula in your program when you do this.) Again, compute an approximate 95% confidence interval for the option price. How does it compare with your answer in (a)?

   (c) How might you further improve the efficiency of the estimator in (b) using antithetic variables? Is it clear that using antithetic variables would be more efficient?

   (d) How might you improve the efficiency of the estimator in (b) using control variables?

3. **(Pricing a down-and-in call option with a discretely monitored barrier)**
   Let $0 = t_0 < t_1 < \cdots < t_m = T$ be the monitoring instants and S_{t_i} the price of the underlying security at the i^{th} such instant. We assume as usual that $S_t \sim \mathrm{GBM}(r, \sigma)$ under the risk-neutral probability measure, Q, and where r is the risk-free continuously compounded rate of interest earned on the cash account. Then the option price is given by

   $$C_0 = \mathbb{E}_0^Q \left[e^{-rT} \max(S_T - K,\, 0) \mathbf{1}_{\{\tau_H \leq T\}} \right]$$

   where H is the barrier and, τ_H is the first monitoring time at which the barrier is breached and $\mathbf{1}_{\{\tau_H \leq T\}}$ is the indicator function of the event that $\tau_H \leq T$.

   Design a conditional Monte-Carlo simulation algorithm to estimate C_0 that conditions on $\{S_0, \ldots, S_{\tau_H}\}$. You should provide pseudo-code for your algorithm and then describe in words what your algorithm actually does.

4. Let the lifetime of a particular piece of machinery be given by X and suppose that X has an exponential distribution with mean 1. We would like to estimate θ using simulation where $\theta := \mathrm{P}(X > 20)$.

   (a) Use 10,000 samples to estimate θ without using any variance reduction techniques. What is the estimated variance of this estimator? Compare your estimated variance with the true variance of the estimator. (This can be computed easily in terms of θ which can itself be easily computed.)

(b) Now consider using importance sampling to estimate θ where we sample from an exponential density function with a different mean, λ. What would be a good choice of λ? Justify your answer.

(c) Now estimate θ using 10,000 samples and the importance sampling density you chose in (b). Estimate the variance of your estimator and compare it to your answers in (a) and (b).

5. Suppose we have a "black box" which on command can generate the value of a Gamma random variable with parameters 3/2 and 1. Explain how we can use this black box to approximate $\theta := \mathbb{E}[e^{-X}/(X+1)]$ where X is an exponential random variable with mean 1.

6. Could you use importance sampling to significantly improve your simulation-based estimate of a deep-in-the-money European call option on some underlying security? (Hint: Use put-call parity.)

7. This question explains how importance sampling can be extremely useful for rare-event simulation. Consider the problem of estimating

$$\theta(x) = \mathbb{E}[e^{\sqrt{Z}} I_{\{Z \geq x\}}]$$

for $x \geq 1$ and where $Z \sim \mathrm{N}(0,1)$. Let $X := e^{\sqrt{Z}} I_{\{Z \geq x\}}$.

(a) Show that $\theta(x) \geq (1 - \Phi(x))e^{\sqrt{x}}$ where $\Phi(.)$ is the CDF of a standard normal random variable.

(b) Show that $\mathbb{E}[X^2] \geq (1 - \Phi(x))e^{2\sqrt{x}}$.

(c) Consider doing importance sampling with a new random variable $W \sim \mathrm{N}(\mu_x, 1)$. (That is, we shift the mean by μ_x, where μ_x depends on x.) Show that the application of the 'maximum principle' yields $\mu_x = x$.

(d) Now set $\mu_x = x$. Use the importance sampling principle to show that $\theta = \mathbb{E}[Y]$ where

$$Y = e^{x^2/2} e^{\sqrt{W}} e^{-Wx} I_{\{W > x\}}.$$

Show also that $\mathbb{E}[Y] \leq e^{-x^2/2}\mathbb{E}[e^{\sqrt{W}} I_{\{W>x\}}]$.

(e) Using the fact that W may be expressed as $Z + x$, show that

$$\theta(x) \leq e^{1-(x-1)^2/2}.$$

Hence show that $\theta(x) \to 0$ as $x \to \infty$.

(f) We will now develop an upper bound for the second moment using importance sampling. Show that

$$\mathbb{E}[Y^2] \leq e^{-(x-1)^2} e^3.$$

(g) Using the above results, show that

$$\frac{\mathrm{Var}(X)}{\mathrm{Var}(Y)} \to \infty$$

as $x \to \infty$. (You should use L'Hôpitals rule for computing the limit.) This means that the improvement factor of the importance sampling algorithm over the standard simulation algorithm tends to infinity as the event becomes rarer.