IEOR E4703 Monte-Carlo Simulation
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Due: 5.00pm Friday 10 February 2017

Assignment 2

1. **(From *Simulation* by Sheldon M. Ross)**
   Show that in estimating $\theta := \mathbb{E}\left[(1-U^2)^{1/2}\right]$ it is better to use U^2 rather than U as a control variate. To do this, use simulation to approximate the necessary covariances. (You may assume $U \sim U(0,1)$.)

2. Referring to Question 1, we will estimate using two covariates, U and U^2. First, use a pilot study of 1000 samples to estimate the optimal values of c_1 and c_2. Then estimate θ using 10,000 samples and the two covariates, i.e., the estimator we use is given by

$$\hat{\theta}_{c,n} = \hat{\theta}_n + c_1\left(\overline{U} - \mathbb{E}[U]\right) + c_2\left(\overline{U^2} - \mathbb{E}[U^2]\right)$$

   where $n = 10,000$ and $\hat{\theta}_n$ is the naive estimator of θ that does not use control variates. Estimate the variance of this estimator and compare it with the variance of the estimator in Question 1 that uses the single control variate, U^2.

3. Consider the problem of pricing an Asian call option as described in Question 10 of Assignment 1.

   (a) Modify your program to compute approximate 99% confidence intervals for the option price for each value of K.

   (b) Show that

$$\mathbb{E}\left[\frac{\sum_{i=1}^{m} S_{iT/m}}{m}\right] = S_0\,\frac{\sum_{i=1}^{m} e^{riT/m}}{m}$$

   (c) Why might the quantity inside the expectation of part (b) be a good control variate?

   (d) Use 1000 replications for the pilot runs and then 10,000 runs for the actual simulation to estimate c and the option price, respectively, for the different values of K.

   (e) For each value of K, compute a new approximate 99% confidence interval for the option price using the control variate estimator. How do these confidence intervals compare to those computed in part (a)?

4. The Asian option example of the previous question provides a very famous example of a control variate. This control variate is based on the the *geometric mean* of $S_{iT/m}$, for $i = 1, \ldots, m$. Define such a control variate and give an analytic expression for its expectation.

5. **(Variation on an exercise from *Stochastic Simulation* by Asmussen and Glynn)**
   Consider a European call option with a maturity of $T = 3$ years, strike price K, underlying asset price process S_t with $S_0 = 100$ and risk-free interest rate 4%. It is assumed that S_t evolves like GBM but with stochastic volatility σ_t, such that σ_t is Markov with two states $\sigma = 0.25$ (the baseline volatility) and $\sigma = 0.75$, and switching intensities $\lambda_+ = 1$, $\lambda_- = 3$ for transitions $0.25 \to 0.75$ and $0.75 \to 0.25$, respectively. It is easy to see that the risk-neutral dynamics are given by

$$dX_t = (r - \sigma_t^2/2)\,dt + \sigma_t dW_t \tag{1}$$

where $X_t := \log S_t - \log S_0$ and W_t is a standard Brownian motion.

Give simulation estimates of the option price for $K = 50$, 100, 150 using

$$S_T,\ Y_T,\ Y_T^2,\ e^{Y_T},\ [S_0 e^{Y_T} - K]^+$$

where Y_t satisfies $Y_0 = 0$ and

$$dY_t = (r - \sigma^2/2)\, dt + \sigma dW_t$$

where W_t is the same driving Brownian motion as in (1). Use both single and multiple control variates and report on the variance reduction for various subsets of the controls.

Note that we can simulate the stochastic volatility process by simulating exponential random variables with mean $1/\lambda_+$ or $1/\lambda_-$ as appropriate. And of course it's possible that multiple transitions could occur during the lifetime of the option.

6. **(From *Simulation* by Sheldon M. Ross)**
   Let X_i, $i = 1, \ldots, 5$, be independent exponential random variables each with mean 1, and consider the quantity θ defined by

   $$\theta := \mathrm{P}\left(\sum_{i=1}^{5} iX_i \geq 21.6\right).$$

   (a) Explain how we can use simulation in this case to estimate θ.

   (b) Give the antithetic variable estimator.

   (c) Is the use of antithetic variables efficient in this case?

7. Estimate θ of Question 6 using both the usual / naive simulation algorithm and using antithetic variates. What sort of variance reduction, if any, do you obtain?

8. **(From *Simulation* by Sheldon M. Ross)**

   (a) If Z is a unit normal random variable, design a study using antithetic variables to estimate $\theta := \mathbb{E}[Z^3 e^Z]$.

   (b) Using the above, do the simulation to obtain an interval of length no greater than 0.1 that you can assert, with 95 percent confidence, contains the value θ.