IEOR E4703 Monte-Carlo Simulation
Martin Haugh

Due: 12.55pm Tuesday 31 January 2017

Assignment 1

1. Use simulation to approximate $\theta := \mathrm{Cov}(U, e^U)$, where U is uniform on $(0,1)$. Compare your approximation with the exact answer if you can.

2. **(Law and Kelton, Q7.13).**
Suppose that $U_1, U_2, \ldots, U_k$ are IID $U(0,1)$ random variables. Show that the fractional part, i.e., ignoring anything to the left of the decimal point of $U_1 + U_2 + \cdots + U_k$ is also uniformly distributed on $(0,1)$.

3. Give the inverse transform algorithm to generate a random variate with the standard right-triangular distribution, i.e., the distribution with PDF
$$f(x) = 2(1-x), \quad 0 \le x \le 1.$$

4. The double exponential distribution has density
$$f(x) = \frac{1}{2}\,\mathbf{1}_{(-\infty,0)}(x)e^{x} + \frac{1}{2}\,\mathbf{1}_{[0,\infty)}(x)e^{-x}$$
Show how to simulate a random variable with density f using the inverse transform method

5. A deck of 100 cards, numbered $1, 2, \ldots, 100$, are shuffled and then turned over one at a time. Say that a 'hit' occurs whenever card i is the i^{th} to be turned over, $i = 1, 2, \ldots, 100$. Write a simulation program to estimate the expectation and variance of the total number of hits. Run the program. Find the exact answers and compare them with your estimates.

6. **(From *Simulation* by Sheldon M. Ross)**
Give a method to generate a random variable having distribution function
$$F(x) = \int_0^\infty x^y e^{-y}\, dy, \quad 0 \le x \le 1$$
and prove that your algorithm is correct. *Hint*: Think in terms of the composition method.

7. **(From *Simulation* by Sheldon M. Ross)**
Let G be a distribution function with density g and suppose, for constants $a < b$, we want to generate a random variable from the distribution function
$$F(x) = \frac{G(x) - G(a)}{G(b) - G(a)}, \quad a \le x \le b.$$
   (a) If X has distribution G, then F is the conditional distribution of X given what information?
   (b) Show that the rejection method reduces in this case to generating a random variable X having distribution G and then accepting it if it lies between a and b.

8. Let (V_1, V_2), in Cartesian coordinates, be uniformly distributed in a circle centered at the origin and with radius R. Show that V_1/V_2 has a Cauchy distribution, i.e.,

$$\mathrm{P}\left(\frac{V_1}{V_2} \leq x\right) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}.$$

Use this fact to construct an algorithm to generate Cauchy variables. (Recall: For any real number, x, $\arctan(x)$ is defined to be the $\theta \in [-\pi/2, \pi/2]$ such that $\tan(\theta) = x$. Such a θ exists and is unique.)

9. Suppose a store opens for business between $t = 0$ and $t = 10$ and that arrivals to the store during $[0, 10]$ constitute a non-homogeneous Poisson process with intensity function $\lambda(t) = (1 + 2t + t^2)/100$. (The fact that the intensity function is increasing might reflect the fact that rush hour occurs at the end of the time period.) Each arrival is equally likely to spend 100, 500 or 1000.

   (a) Use the thinning algorithm to simulate arrivals to the store and to estimate the average amount of money that is spent in $[0, 10]$. (You should simulate $10,000$ periods to obtain your estimate.)

   (b) Now compute analytically the expected amount of money that is spent in $[0, 10]$ and compare it to your estimate.

10. Use simulation to estimate the price of an Asian call option where the time T payoff is given by

$$h(\mathbf{X}) := \max\left(0, \frac{\sum_{i=1}^{m} S_{iT/m}}{m} - K\right)$$

where $\mathbf{X} = (S_{T/m}, S_{2T/m}, \ldots, S_T)$. You may assume that $S_t \sim \mathrm{GBM}(r.\sigma)$ under the risk-neutral probability measure where $r = .05$ and $\sigma = .25$. Other relevant parameters are $T = 1$ year, $S_0 = 100$ and $m = 11$. Use $10,000$ simulations and estimate the option price for $K = 90, 100, 110$ and 120.

11. Consider a standard Brownian motion, B_t, and suppose we want to sample the Brownian motion at the following times $t_1 < t_2 < \cdots < t_n$.

   (a) Write out the variance - covariance matrix, $\mathbf{\Sigma}$, of $\mathbf{X} := (B_{t_1}, \ldots, B_{t_n})$.

   (b) Find the Cholesky decomposition of $\mathbf{\Sigma}$. (One way to do this is to review or learn some linear algebra and do the decomposition directly. Another way to do it is to guess what the decomposition is and then show that your guess is indeed correct. To come up with a good guess, you might want to compute the Cholesky decomposition of $\mathbf{\Sigma}$ for some trial values of $(t_1, t_2, \ldots, t_n)$.)

   (c) Since $\mathbf{X}$ has a multivariate normal distribution, one method of simulating $\mathbf{X}$ is to generate an n-vector, $\mathbf{Z}$, of IID standard normal random variables and use the Cholesky decomposition to convert $\mathbf{Z}$ into a sample value of $\mathbf{X}$. Given your response to (b), is this really any different to the method we described in class for simulating a Brownian motion? Explain your answer.