Foundations of Financial Engineering

A Very Brief Introduction to Stochastic Calculus

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A Very Brief Introduction to Stochastic Calculus

There is a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- $\mathbb{P}$ is the "true" or *physical* probability measure.
- Ω is the universe of possible outcomes.

  Use $\omega \in \Omega$ to represent a generic outcome, typically a sample path of a stochastic process.
- $\mathcal{F}$ represents the set of possible *events* where an event is a subset of Ω.

There is also a filtration, $\{\mathcal{F}_t\}_{t \geq 0}$, that models the evolution of information through time

- e.g. if known by time t whether or not event E has occurred, then $E \in \mathcal{F}_t$
- if working with a finite horizon, $[0, T]$, then can take $\mathcal{F} = \mathcal{F}_T$.

Also say that a stochastic process, X_t, is $\mathcal{F}_t$-adapted if any events that depend only on $\{X_s\}_{0 \leq s \leq t}$ are in $\mathcal{F}_t$.

Martingales and Brownian Motion

Definition: A stochastic process, $\{ W_t \ : \ 0 \leq t \leq \infty \}$, is a standard Brownian motion if:

1. $W_0 = 0$
2. It has continuous sample paths
3. It has independent, stationary increments.
4. $W_t \sim \mathsf{N}(0, t)$.

Definition: An n-dimensional process, $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$, is a standard n-dimensional Brownian motion if each $W_t^{(i)}$ is a standard Brownian motion and the $W_t^{(i)}$'s are independent of each other.

Martingales and Brownian Motion

Definition: A stochastic process, $\{X_t \ : \ 0 \leq t \leq \infty\}$, is a martingale with respect to the filtration, $\mathcal{F}_t$, and probability measure, P, if

1. $\mathsf{E}^P[|X_t|] < \infty$ for all $t \geq 0$
2. $\mathsf{E}^P[X_{t+s}|\mathcal{F}_t] = X_t$ for all $t, s \geq 0$.

Example: Let W_t be a Brownian motion. Then the following are all martingales:

1. W_t

2. $W_t^2 - t$

3. $\exp\left(\theta W_t - \theta^2 t/2\right)$

Brownian martingales

$M_t := \exp(\theta W_t - \theta^2 t/2)$ is an example of an exponential martingale

- of particular significance since they are positive and therefore may be used to define new probability measures.

Example (Doob or Levy Martingale):
Let Z be a random variable and set $X_t := \mathsf{E}[Z|\mathcal{F}_t]$. Then X_t is a martingale.

Quadratic Variation

Consider a partition of the time interval, $[0, T]$ given by

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T.$$

Let X_t be a stochastic process and consider the sum of squared changes

$$Q_n(T) := \sum_{i=1}^{n} \left[\Delta X_{t_i}\right]^2$$

where $\Delta X_{t_i} := X_{t_i} - X_{t_{i-1}}$.

Definition: The quadratic variation of a stochastic process, X_t, is equal to the limit of $Q_n(T)$ as $\Delta t := \max_i(t_i - t_{i-1}) \to 0$.

The functions with which you are normally familiar, e.g. continuous differentiable functions, have quadratic variation equal to zero.

Quadratic Variation

Theorem: The quadratic variation of a Brownian motion is equal to T with probability 1.

Total Variation

Definition: The total variation of a process, X_t, on $[0, T]$ is defined as

$$\text{Total Variation} := \lim_{\Delta t \to 0} \sum_{i=1}^{n} |X_{t_k} - X_{t_{k-1}}|.$$

Note that any continuous stochastic process or function that has non-zero quadratic variation must have infinite total variation.

Follows by observing that

$$\sum_{i=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \leq \left(\sum_{i=1}^{n} |X_{t_k} - X_{t_{k-1}}| \right) \max_{1 \leq k \leq n} |X_{t_k} - X_{t_{k-1}}|. \quad (1)$$

Now let $n \to \infty$ in (1) then the continuity of X_t implies the result.

Therefore follows that the total variation of a Brownian motion is infinite.

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Stochastic Integrals

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Stochastic Integrals

Now write $X_t(\omega)$ instead of usual X_t to emphasize that the quantities in question are stochastic.

Definition: A stopping time of the filtration $\mathcal{F}_t$ is a random time, τ, such that the event $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t > 0$.

Definition: We say a process, $h_t(\omega)$, is elementary if it is piece-wise constant so there exists a sequence of stopping times $0 = t_0 < t_1 < \ \ldots \ < t_n = T$ and a set of $\mathcal{F}_{t_i}$-measurable functions, $e_i(\omega)$, such that

$$h_t(\omega) \; = \; \sum_i e_i(\omega) I_{[t_i, t_{i+1})}(t)$$

where $I_{[t_i, t_{i+1})}(t) = 1$ if $t \in [t_i, t_{i+1})$ and 0 otherwise.

Stochastic Integrals

Definition: The stochastic integral of an elementary function, $h_t(\omega)$, with respect to a Brownian motion, W_t, is defined as

$$\int_0^T h_t(\omega)\ dW_t(\omega) \ := \ \sum_{i=0}^{n-1} e_i(\omega) \left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega) \right). \tag{2}$$

Note that our definition of an elementary function assumes that the function, $h_t(\omega)$, is evaluated at the left-hand point of the interval in which t falls

- a key component in the definition of the stochastic integral
- without it many later results would no longer hold.
- moreover, defining the stochastic integral in this way makes the resulting theory suitable for financial applications.

For a more general process, $X_t(\omega)$, we have

$$\int_0^T X_t(\omega)\ dW_t(\omega) \ := \ \lim_{n\to\infty} \int_0^T X_t^{(n)}(\omega)\ dW_t(\omega)$$

where $X_t^{(n)}$ is a sequence of elementary processes that "converges" to X_t.

Computing $\int_0^T W_t \, dW_t$

Example: Let $0 = t_0^n < t_1^n < t_2^n < \ldots < t_n^n = T$ be a partition of $[0, T]$ and define

$$X_t^n \; := \; \sum_{i=0}^{n-1} W_{t_i^n} I_{[t_i^n, t_{i+1}^n)}(t)$$

where $I_{[t_i^n, t_{i+1}^n)}(t) = 1$ if $t \in [t_i^n, t_{i+1}^n)$ and is 0 otherwise.

Then X_t^n is an adapted elementary process and, by continuity of Brownian motion, satisfies $\lim_{n \to \infty} X_t^n = W_t$ almost surely as $\max_i |t_{i+1}^n - t_i^n| \to 0$.

Computing $\int_0^T W_t \, dW_t$

By (2) the stochastic integral of X_t^n is:

$$\begin{aligned}
\int_0^T X_t^n \, dW_t &= \sum_{i=0}^{n-1} W_{t_i^n} (W_{t_{i+1}^n} - W_{t_i^n}) \\
&= \frac{1}{2} \sum_{i=0}^{n-1} \left(W_{t_{i+1}^n}^2 - W_{t_i^n}^2 - (W_{t_{i+1}^n} - W_{t_i^n})^2 \right) \\
&= \frac{1}{2} W_T^2 - \frac{1}{2} W_0^2 - \frac{1}{2} \sum_{i=0}^{n-1} (W_{t_{i+1}^n} - W_{t_i^n})^2. \qquad (3)
\end{aligned}$$

Therefore obtain

$$\int_0^T W_t \, dW_t = \lim_{n \to \infty} \int_0^T X_t^n \, dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T.$$

We generally evaluate stochastic integrals using Itô's Lemma.

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Itô's Isometry and the Martingale Property of Stochastic Integrals

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Itô's Isometry

Definition: We define the space $L^2[0,T]$ to be the space of processes, $X_t(\omega)$, such that

$$\mathsf{E}\left[\int_0^T X_t(\omega)^2 \; dt\right] < \infty.$$

Theorem: (Itô's Isometry) For any $X_t(\omega) \in L^2[0,T]$ we have

$$\mathsf{E}\left[\left(\int_0^T X_t(\omega) \; dW_t(\omega)\right)^2\right] = \mathsf{E}\left[\int_0^T X_t(\omega)^2 \; dt\right].$$

Proof: (When X_t is an elementary process)

Let $X_t = \sum_i e_i(\omega) I_{[t_i, t_{i+1})}(t)$ be an elementary process.

Therefore have $\int_0^T X_t(\omega) \; dW_t(\omega) \; := \; \sum_{i=0}^{n-1} e_i(\omega) \left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)$ so that:

Proof of Itô's Isometry When X_t is Elementary

$$
\begin{aligned}
\mathsf{E}\left[\left(\int_0^T X_t(\omega)\ dW_t(\omega)\right)^2\right] &= \mathsf{E}\left[\left(\sum_{i=0}^{n-1} e_i(\omega)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)\right)^2\right] \\
&= \sum_{i=0}^{n-1} \mathsf{E}\left[e_i^2(\omega)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)^2\right] \\
&\quad + 2 \sum_{0 \le i < j \le n-1}^{n-1} \mathsf{E}\left[e_i(\omega)\ e_j(\omega)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)\left(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right)\right]
\end{aligned}
$$

Proof of Itô's Isometry When X_t is Elementary

$$= \sum_{i=0}^{n-1} \mathsf{E}\left[e_i^2(\omega)\ \underbrace{\mathsf{E}_{t_i}\left[\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)^2\right]}_{=\,t_{i+1}-t_i} \right]$$

$$+\ 2 \sum_{0\leq i<j\leq n-1}^{n-1} \mathsf{E}\left[e_i(\omega)\ e_j(\omega) \left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right)\ \underbrace{\mathsf{E}_{t_j}\left[\left(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right)\right]}_{=0} \right]$$

$$= \mathsf{E}\left[\sum_{i=0}^{n-1} e_i^2(\omega)(t_{i+1} - t_i)\right]$$

$$= \mathsf{E}\left[\int_0^T X_t(\omega)^2\ dt\right]$$

as required.

Martingale Property of Stochastic Integrals

Theorem: The stochastic integral, $Y_t := \int_0^t X_s(\omega)\ dW_s(\omega)$, is a martingale for any $X_t(\omega) \in L^2[0, T]$.

This theorem is known as the martingale property of stochastic integrals

- a very important result.

Easy to prove when X_t is an elementary process.

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Stochastic Differential Equations and Itô's Lemma

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Stochastic Differential Equations

Definition: An n-dimensional Itô process, X_t, is a process that can be represented as

$$X_t \;=\; X_0 + \int_0^t a_s \; ds + \int_0^t b_s \; dW_s \tag{4}$$

where W is an m-dimensional standard Brownian motion, and a and b are n-dimensional and $n \times m$-dimensional $\mathcal{F}_t$-adapted processes, respectively.

Often use notation $dX_t \;=\; a_t\, dt + b_t\, dW_t$ as shorthand for (4).

An n-dimensional stochastic differential equation (SDE) has the form

$$dX_t \;=\; a(X_t, t) \; dt + b(X_t, t) \; dW_t; \quad X_0 = x. \tag{5}$$

Once again, (5) is shorthand for

$$X_t \;=\; x + \int_0^t a(X_s, s) \; dt + \int_0^t b(X_s, t) \; dW_s. \tag{6}$$

Conditions exist to guarantee existence and uniqueness of solutions to (6).

Itô's Lemma

A useful tool for solving SDE's is Itô's Lemma, probably the most important result in stochastic calculus!

Theorem: (Itô's Lemma for 1-dimensional Brownian Motion)
Let W_t be a Brownian motion on $[0, T]$ and suppose $f(x)$ is a twice continuously differentiable function on $\mathbf{R}$.

Then for any $t \leq T$ we have

$$f(W_t) = f(0) + \frac{1}{2} \int_0^t f''(W_s)\, ds + \int_0^t f'(W_s)\, dW_s. \tag{7}$$

Sketch Proof of Itô's Lemma

Proof Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = t$ be a partition of $[0, t]$. Then

$$f(W_t) \;=\; f(0) \;+\; \sum_{i=0}^{n-1} \left(f(W_{t_{i+1}}) - f(W_{t_i}) \right). \tag{8}$$

Taylor's Theorem implies

$$f(W_{t_{i+1}}) - f(W_{t_i}) \;=\; f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) \;+\; \frac{1}{2} f''(\theta_i)(W_{t_{i+1}} - W_{t_i})^2 \tag{9}$$

for some $\theta_i \in (W_{t_i}, W_{t_{i+1}})$.

Substituting (9) into (8) obtain

$$f(W_t) \;=\; f(0) + \sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(\theta_i)(W_{t_{i+1}} - W_{t_i})^2. \tag{10}$$

If $\delta := \max_i |t_{i+1} - t_i| \to 0$ then can be shown that terms on rhs of (10) converge to corresponding terms on rhs of (7) as desired

- not surprising since quadratic variation of Brownian motion on $[0, t] = t$. $\square$

Itô's Lemma

A more general version of Itô's Lemma can be stated for Itô processes.

Theorem. (Itô's Lemma for 1-dimensional Itô process)
Let X_t be a 1-dimensional Itô process satisfying the SDE

$$dX_t \;=\; \mu_t \, dt + \sigma_t \, dW_t.$$

If $f(t,x) : [0,\infty) \times R \to R$ is a $C^{1,2}$ function and $Z_t := f(t, X_t)$ then

$$\begin{aligned} dZ_t \quad &= \quad \frac{\partial f}{\partial t}(t, X_t) \; dt + \frac{\partial f}{\partial x}(t, X_t) \; dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \; (dX_t)^2 \\ &= \quad \left(\frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) \; \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \; \sigma_t^2 \right) \; dt \; + \; \frac{\partial f}{\partial x}(t, X_t) \; \sigma_t \; dW_t. \end{aligned}$$

The "Box" Calculus

In statement of Itô's Lemma, implicitly assumed that $(dX_t)^2 = \sigma_t^2 \ dt$.

The "box calculus" is a series of simple rules for calculating such quantities:

$$
\begin{aligned}
dt \times dt \; = \; dt \times dW_t \quad &= \quad 0 \quad \text{and} \\
dW_t \times dW_t \quad &= \quad dt
\end{aligned}
$$

When we have two correlated Brownian motions, $W_t^{(1)}$ and $W_t^{(2)}$, with correlation coefficient, ρ, then we easily obtain that $dW_t^{(1)} \times dW_t^{(2)} = \rho \ dt$.

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Some Examples of Itô's Lemma in Action

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Geometric Brownian Motion

Example: Suppose a stock price, S_t, satisfies the SDE

$$dS_t \;=\; \mu_t S_t \; dt + \sigma_t S_t \; dW_t.$$

Then can use substitution, $Y_t = \log(S_t)$ and Itô's Lemma applied to the function $f(x) := \log(x)$ to obtain

$$S_t \;=\; S_0 \exp\left(\int_0^t (\mu_s - \sigma_s^2/2)\; ds + \int_0^t \sigma_s \; dW_s\right). \tag{11}$$

Geometric Brownian Motion

Note S_t does not appear on rhs of (11) so that we have indeed solved the SDE.

When $\mu_s = \mu$ and $\sigma_s = \sigma$ we obtain

$$S_t \;=\; S_0 \exp\left((\mu - \sigma^2/2)\; t + \sigma W_t\right) \tag{12}$$

so that $\log(S_t) \sim \mathsf{N}\left((\mu - \sigma^2/2)t,\; \sigma^2 t\right)$.

In this case we say S_t is a geometric Brownian motion (GBM).

The Ornstein-Uhlenbeck Process

Example: Let S_t be a security price and suppose $X_t = \log(S_t)$ satisfies the SDE

$$dX_t \ = \ [-\gamma(X_t - \mu t) + \mu] \ dt \ + \ \sigma \, dW_t.$$

We would like to solve this SDE.

So first recall Itô's Lemma: If $Z_t := f(t, X_t)$ where $dX_t = \mu_t \, dt + \sigma_t \, dW_t$ then

$$dZ_t \ = \ \left(\frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) \ \mu_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \ \sigma_t^2 \right) \ dt \ + \ \frac{\partial f}{\partial x}(t, X_t) \ \sigma_t \ dW_t.$$

So let's apply Itô's Lemma to $Z_t := e^{\gamma t} \, X_t$:

The Ornstein-Uhlenbeck Process

We obtain

$$\begin{aligned} dZ_t &= \left(\gamma e^{\gamma t} X_t + e^{\gamma t}[-\gamma(X_t - \mu t) + \mu]\right) dt + e^{\gamma t}\sigma\, dW_t \\ &= \mu e^{\gamma t}\, (\gamma t + 1)\; dt + e^{\gamma t}\sigma\, dW_t \end{aligned}$$

so that

$$Z_t \;=\; Z_0 \;+\; \mu \int_0^t e^{\gamma s}\, (\gamma s + 1)\;\, ds \;+\; \sigma \int_0^t e^{\gamma s}\; dW_s$$

After simplifying we have:

$$X_t = X_0\, e^{-\gamma t} + \mu t + \sigma\, e^{-\gamma t} \int_0^t e^{\gamma s}\;\, dW_s. \tag{13}$$

Note that X_t does not appear on rhs of (13) so that we have solved the SDE!

The Ornstein-Uhlenbeck Process

We also obtain:

$$\mathsf{E}[X_t] \;=\; X_0 e^{-\gamma t} \;+\; \mu t$$

and

$$\begin{aligned}
\mathsf{Var}(X_t) &= \mathsf{Var}\left(\sigma e^{-\gamma t} \int_0^t e^{\gamma s}\, dW_s\right) \\
&= \sigma^2 e^{-2\gamma t}\, \mathsf{E}\left[\left(\int_0^t e^{\gamma s}\, dW_s\right)^2\right] \\
&= \sigma^2 e^{-2\gamma t} \int_0^t e^{2\gamma s}\, ds \\
&= \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right).
\end{aligned}$$

Question: What is the distribution of S_T?