Foundations of Financial Engineering

Introduction to Real Options

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Introduction to Real Options

Principal characteristics shared by real options problems:

1. They involve non-financial assets, e.g. factory capacity, oil leases, commodities, technology from R&D etc.

   Often the case, however, that financial uncertainty is also present.

2. Incomplete markets – as stochastic processes driving non-financial variables will not be "spanned" by the financial assets.

   e.g. Not possible to construct a self-financing trading strategy that replicates a payoff whose value depends on whether or not there is oil in a particular oilfield, or whether or not a particular manufacturing product will be popular with consumers.

   Therefore use economics to guide us in choosing a good EMM (or set of EMM's) to price the real options.

3. There are usually options available to the decision-maker.

   More generally, real options problems are usually control problems where the decision-maker can (partially) control some of the quantities under consideration.

A Real Options Example: the Simplico Gold Mine

Gold Price Lattice

										2063.9
									1719.9	1547.9
								1433.3	1289.9	1161.0
							1194.4	1075.0	967.5	870.7
						995.3	895.8	806.2	725.6	653.0
					829.4	746.5	671.8	604.7	544.2	489.8
				691.2	622.1	559.9	503.9	453.5	408.1	367.3
			576.0	518.4	466.6	419.9	377.9	340.1	306.1	275.5
		480.0	432.0	388.8	349.9	314.9	283.4	255.1	229.6	206.6
	400.0	360.0	324.0	291.6	262.4	236.2	212.6	191.3	172.2	155.0
Date	0	1	2	3	4	5	6	7	8	9

Current market price of gold is $\$400$ and it follows a binomial model:

- it increases each year by a factor of 1.2 with probability $.75$
- or it decreases by a factor of $.9$ with probability $.25$.

Interest rates are flat at $r = 10\%$ per year.

Luenberger's Simplico Gold Mine

Gold can be extracted from the Simplico gold mine at a rate of up to $10,000$ ounces per year at a cost of $C = \$200$ per ounce.

Want to compute the price of a lease on the mine that expires after 10 years.

Any gold that is extracted in a given year is sold at the end of the year at the price that prevailed at the beginning of the year.

Gold is a traded commodity so we can obtain a unique risk-neutral price for any derivative security dependent upon its price process

- risk-neutral probabilities are found to be $q = 2/3$ and $1 - q = 1/3$.

Value of lease is then computed by working backwards in the lattice below

- because the lease expires worthless the node values at $t = 10$ are all zero.

A Real Options Example: the Simplico Gold Mine

Lease Value (in millions)

										16.9
									27.8	12.3
								34.1	20.0	8.7
							37.1	24.3	14.1	6.1
						37.7	26.2	17.0	9.7	4.1
					36.5	26.4	18.1	11.5	6.4	2.6
				34.2	25.2	17.9	12.0	7.4	3.9	1.5
			31.2	23.3	16.7	11.5	7.4	4.3	2.1	0.7
		27.8	20.7	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.1	17.9	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

e.g. Value of 16.9 on uppermost node at $t = 9$ is obtained by discounting the profits earned at $t = 10$ back to the beginning of the year:

$$16.94\text{m} = 10\text{k} \times (2,063.9 - 200)/1.1.$$

A Real Options Example: the Simplico Gold Mine

Lease Value (in millions)

										16.9
									27.8	12.3
								34.1	20.0	8.7
							37.1	24.3	14.1	6.1
						37.7	26.2	17.0	9.7	4.1
					36.5	26.4	18.1	11.5	6.4	2.6
				34.2	25.2	17.9	12.0	7.4	3.9	1.5
			31.2	23.3	16.7	11.5	7.4	4.3	2.1	0.7
		27.8	20.7	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.1	17.9	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Node value in any earlier year obtained by summing together discounted expected value of lease and profit (obtained at the end of year) back to beginning of year.

e.g. In year 6 central node has a value of 12 million because:

$$12.0\text{m} \;=\; \frac{10\text{k} \times (503.9 - 200)}{1.1} \;+\; \frac{q \times 11.5\text{m} + (1-q) \times 7.4\text{m}}{1.1}.$$

Luenberger's Simplico Gold Mine

Lease Value (in millions)

										16.9
									27.8	12.3
								34.1	20.0	8.7
							37.1	24.3	14.1	6.1
						37.7	26.2	17.0	9.7	4.1
					36.5	26.4	18.1	11.5	6.4	2.6
				34.2	25.2	17.9	12.0	7.4	3.9	1.5
			31.2	23.3	16.7	11.5	7.4	4.3	2.1	0.7
		27.8	20.7	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.1	17.9	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Of course never optimal to extract gold when price less than $\$200$.

Backwards evaluation therefore takes the form

$$V_t(s) = \frac{10\text{k} \times \max\{0, s - C\} + (q V_{t+1}(us) + (1-q) V_{t+1}(ds))}{1+r}$$

where $V_t(s) =$ time t value of lease when gold price is s and $C = 200$.

Foundations of Financial Engineering

Should We Enhance the Simplico Goldmine?

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Should We Enhance the Simplico Goldmine?

Suppose it's possible to enhance extraction rate to $12,500$ ounces per year by purchasing new equipment that costs $\$4$ million.

Once new equipment in place then it remains in place for all future years.

Moreover the extraction cost would also increase to $\$240$ per ounce with the enhancement in place.

At end of lease new equipment becomes property of the original owner of mine.

Owner of lease therefore has an option to install the new equipment at any time

- we want to determine value of this option!

To do this, must first compute lease value assuming new equipment is in place at $t = 0$

- done in exactly the same manner as before
- values at each node and period are given in following lattice:

Should We Enhance the Simplico Goldmine?

Lease Value Assuming Enhancement in Place (in millions)

										20.7
									33.9	14.9
								41.4	24.1	10.5
							44.8	29.2	16.8	7.2
						45.2	31.2	20.0	11.3	4.7
					43.5	31.0	21.0	13.2	7.2	2.8
				40.4	29.3	20.4	13.4	8.0	4.1	1.4
			36.4	26.6	18.7	12.5	7.7	4.1	1.8	0.4
		31.8	23.3	16.3	10.8	6.5	3.4	1.3	0.2	0.0
	27.0	19.5	13.5	8.6	4.9	2.3	0.8	0.1	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Backwards evaluation therefore takes the form

$$V_t^{\textsf{eq}}(s) = \frac{12.5\text{k} \times \max\{0, s - C_{new}\} + (qV_{t+1}(us) + (1-q)\,V_{t+1}(ds))}{1+r}$$

where $V_t^{\textsf{eq}}(s) =$ time t value of lease when gold price $= s$ and $C_{new} = 240$.

Should We Enhance the Simplico Goldmine?

So $V_t^{\text{eq}}(s) :=$ time t lease value when gold price $= s$ and new equipment in place

- note the $\$4$ million cost of the new equipment has not been subtracted.

We find $V_0^{\text{eq}}(400) = 27$m.

Now let $U_t(s) :=$ time t price of lease when gold price $= s$ and with the option to enhance in place.

Can then solve for $U_t(s)$ as follows:

$$U_t(s) = \max\left\{ V_t^{\text{eq}}(s) - 4\text{m},\ \frac{10\text{k} \times \max\{0, s - C\} + (qU_{t+1}(us) + (1-q)\,U_{t+1}(ds))}{1+r} \right\}. \tag{1}$$

We want $U_0(s)$ with $s = 400$. Can compute this using (1) as follows:

Should We Enhance the Simplico Goldmine?

1. Construct another lattice that, starting at $t = 10$, assumes new equipment is not in place.
2. Work backwards in lattice, computing lease value at each node as before but now with one added complication:

   After computing lease value, A say, at a node we compare this value to the value, $V_t^{\text{eq}}(s)$, at corresponding node in lattice where enhancement was assumed to be in place.

   If $V_t^{\text{eq}}(s) - 4\text{m} \geq A$ then optimal to install the equipment at this node
   - if it has not already been installed.

   Otherwise not optimal to install lease at this node
   - if it has not already been installed.

   In summary: place $\max(V_t^{\text{eq}}(s) - 4\text{m},\ A)$ at the node in our new lattice.

Continue working backwards using (1), determining at each node whether or not new equipment should be installed if it hasn't been already.

Find lease value with the option is $\$24.6$m
- slightly greater than lease value without the option.

Should We Enhance the Simplico Goldmine?

Lease Value with Option for Enhancement (in millions)

										16.9
									29.9*	12.3
								37.4*	20.1*	8.7
							40.8*	25.2*	14.1	6.1
						41.2*	27.2*	17.0	9.7	4.1
					39.5*	27.0*	18.1	11.5	6.4	2.6
				36.4*	25.6	17.9	12.0	7.4	3.9	1.5
			32.6	23.5	16.7	11.5	7.4	4.3	2.1	0.7
		28.6	20.9	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.6	18.0	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

The equation used to compute the lease value:

$$U_t(s) = \max\left\{ V_t^{\text{eq}}(s) - 4\text{m},\ \frac{10\text{k} \times \max\{0, s - C\} + (qU_{t+1}(us) + (1-q)\,U_{t+1}(ds))}{1+r} \right\}$$

is known as the Bellman equation.

Foundations of Financial Engineering

Zero-Level Pricing with Private Uncertainty

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Zero-Level Pricing with Private Uncertainty

How do we price real options when a unique EMM not available?

Zero-level pricing is one approach

- based on utility maximization for portfolio optimization problems.

The zero-level-price is the price that leaves decision-maker indifferent between purchasing and not purchasing an infinitesimal amount of the security.

We say that a source of uncertainty is private if it is independent of any uncertainty driving the financial markets

- e.g. the success of an R&D project, the quantity of oil in an oilfield, the reliability of a vital piece of manufacturing equipment or the successful launch of a new product
- could also include incidence of natural disasters etc. as sources of "private" uncertainty.

Economic considerations suggest that if we want to use zero-level pricing to compute real option prices when there is only private uncertainty involved, then we should use the true probability distribution to do so and discount by the risk-free interest rate.

Zero-Level Pricing with Private Uncertainty

Some intuition for this observation comes from the CAPM which states

$$\mathsf{E}^{\mathbb{P}}[r_o] \;=\; r_f + \beta_o \left(\mathsf{E}^{\mathbb{P}}[r_m] - r_f \right)$$

where $\beta_o := \mathsf{Cov}(r_m, r_o)/\mathsf{Var}(r_m)$ and r_m is the return on the market portfolio.

If $r_o =$ return on an investment that is only exposed to private uncertainty then $\mathsf{Cov}(r_m, r_o) = 0$ and CAPM implies $\mathsf{E}^{\mathbb{P}}[r_o] = r_f$.

Value of the investment / real option (in a CAPM world) therefore given by

$$P_0 \;=\; \frac{\mathsf{E}^{\mathbb{P}}[P_1]}{1 + r_f}$$

where P_1 is the terminal payoff of the investment.

This and other similar arguments are used to motivate practice of using:

- risk-neutral probabilities to price financial uncertainty of an investment and
- true probabilities to price the non-financial uncertainty.

Note this is consistent with martingale pricing as the use of risk-neutral probabilities ensures that deflated (financial) security prices are martingales.

Zero-Level Pricing with Private Uncertainty

Practice of using true probabilities to price private uncertainty is easier to justify if we can disentangle financial uncertainty from non-financial uncertainty.

Example: Consider a one period model where at $t = 1$ you will have X barrels of oil that you can then sell in the spot oil market for $\$P$ per barrel

- will therefore will earn XP at $t = 1$.

- If X only revealed at $t = 1$, then impossible to fully hedge your oil exposure by trading at $t = 0$.

- But if X revealed at $t = 0$ then can fully hedge resulting oil exposure at $t = 0$ using 1-period oil futures.

  Only non-financial uncertainty (which is revealed at $t = 0$) then remains and so the situation is similar to that in earlier CAPM argument.

Zero-Level Pricing with Private Uncertainty

Reasons for using true probabilities to price non-financial uncertainty, X, is therefore more persuasive when X revealed at $t = 0$.

In former case where X not revealed until $t = 1$, the financial uncertainty cannot be perfectly hedged and the economic argument for using the true probabilities to price the non-financial uncertainty is not as powerful.

Still it can be justified by considering, for example, a first order Taylor expansion of the payoff, XP, about the means, $\bar{X}$ and $\bar{P}$, to obtain

$$\begin{aligned} XP &\approx \bar{X}\bar{P} + \bar{P}(X - \bar{X}) + \bar{X}(P - \bar{P}) \\ &= \bar{P}X + \bar{X}P - \bar{X}\bar{P}. \end{aligned} \tag{2}$$

Note the payoff approximation (2) does allow for the financial and non-financial uncertainties to be disentangled.

Foundations of Financial Engineering

Example: Valuing A Foreign Venture

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Valuing A Foreign Venture

Example: A particular investment gives you the rights to the monthly profits of a foreign venture for a fixed period of time.

The first payment will be made one month from now and the final payment will be in 5 months time after which the investment will be worthless.

The monthly payments are denominated in Euro, and are IID random variables with $\mathbb{P}$-expectation μ.

Payments also independent of returns in both domestic and foreign financial markets.

Would like to determine the value of this investment!

Valuing A Foreign Venture

Let us first assume that the domestic, i.e. US, interest rate is 5% per annum, compounded monthly

- implies a gross rate of 1.0042 per month.

Annual interest rate in Euro zone, i.e. the foreign interest rate, is 10%, again compounded monthly

- implies a per month gross interest rate of 1.0083.

Can construct a binomial lattice for the $/Euro exchange rate process if we view the foreign currency, i.e. the Euro, as an asset that pays "dividends", i.e. interest, in each period.

Martingale pricing in a binomial model for the exchange rate with up- and down-factors, u and d respectively, implies that

$$X_i \quad = \quad \mathsf{E}_i^{\mathbb{Q}}\left[\frac{X_{i+1} + r_f X_i}{1 + r_d}\right] \tag{3}$$

where $X_i = \$/\text{Euro}$ exchange rate at time i.

Valuing A Foreign Venture

Risk-neutral probability, q, of an up-move satisfies

$$q = \frac{1 + r_d - d - r_f}{u - d} \qquad (4)$$

where r_d and r_f are domestic and foreign per-period interest rates, respectively.

The binomial lattice is given below with $X_0 = 1.20$, $u = 1.05$ and $d = 1/u$.

Dollar/Euro Exchange Rate

					1.53
				1.46	1.39
			1.39	1.32	1.26
		1.32	1.26	1.20	1.14
	1.26	1.20	1.14	1.09	1.04
1.20	1.14	1.09	1.04	0.99	0.94
t=0	t=1	t=2	t=3	t=4	t=5

Valuing A Foreign Venture

Valuing the investment using zero-level pricing (and therefore using the the true probability measure, $\mathbb{P}$, for the non-financial uncertainty) is now straightforward:

- at each time-t node in the lattice we assume there is a cash-flow of μX_t.

These cash-flows are valued as usual by backwards evaluation using the risk-neutral probabilities computed in (4).

Question: Can you see an easy way (that does not require backwards evaluation) to value the cash-flows?

Foundations of Financial Engineering

Example: Operating a Gas-Fired Power Plant

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Operating a Gas-Fired Power Plant

Example A power company is considering renting a two-regime gas-fired power plant over the (discrete) time interval $[0, T]$.

To figure out whether or not this is worthwhile, the company must figure out the plant's optimal operating policy.

At any time t company can operate the plant in "shut-down" mode, low-capacity mode or high-capacity mode.

Relevant details for each of these three operating states are:

- Shut-down mode: plant rent K
- Low capacity mode: output $\underline{Q}$, gas consumption $\underline{H}$, plant rent $\underline{K}$
- High capacity mode: output $\overline{Q}$, gas consumption $\overline{H}$, plant rent $\overline{K}$

If the plant is currently in operation then shutting the plant down will incur a ramp-down cost of C_d.

Similarly, if the plan is currently shut down, then moving it into a low / high production mode will incur a ramp-up cost of C_u.

Operating a Gas-Fired Power Plant

Use $s_t \in \{0 \equiv \text{plant shut-down}, 1 \equiv \text{plant operating}\}$ to denote time t state of the plant.

In each period there are two possible actions:
$a \in \{0 \equiv \text{operate in shut-down mode}, 1 \equiv \text{operate in production mode}\}$.

Let $V_t(s, P, G)$ denote optimal profit of plant over horizon $[t, T]$ when current state is s, current electricity price is P, and current price of gas is G.

Also have following additional notation:

- $c(s, a)$: cost of taking action a when plant in state s
- $u(s, a)$: new state of plant when action a is taken in state s

Specific values are:

$$\begin{aligned} c(0,0) &= K \\ c(0,1) &= C_u + K \\ c(1,0) &= C_d + K \\ c(1,1) &= \min\left\{\overline{K} + G_t\overline{H} - P_t\overline{Q},\ \underline{K} + G_t\underline{H} - P_t\underline{Q}\right\} \end{aligned} \qquad \begin{aligned} u(0,0) &= 0 \\ u(0,1) &= 1 \\ u(1,0) &= 0 \\ u(1,1) &= 1 \end{aligned}$$

Operating a Gas-Fired Power Plant

Note that when the plant is in production mode one has the choice to run it at the low or high capacity mode.

Can now write the dynamic programming / Bellman equation:

$$V_t(s_t, P_t, G_t) = \max_{a \in \{0,1\}} \left\{ -c(s_t, a) + e^{-r\Delta t} \mathsf{E}_t^{\mathbb{Q}} \left[V_{t+1}(u(s_t, a), P_{t+1}, G_{t+1}) \right] \right\} \tag{5}$$

where $\mathsf{E}_t^{\mathbb{Q}}[\cdot]$ denotes usual risk-neutral expectation conditional on all known-information at time t, r is the continuously compounded risk-free rate and Δt is the length of a period.

There are two sources of uncertainty here: price dynamics of electricity and gas.

Straightforward to construct lattices for these prices and solve (5) numerically beginning at time T and working backwards in usual manner.