Foundations of Financial Engineering

Some Motivation for Mean-Variance Analysis

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A Simple Motivating Example

Consider a one-period market with n securities satisfying:

$$\begin{aligned}
\mathsf{E}[R_i] &= \mu, \quad i = 1, \ldots, n \\
\mathsf{Var}(R_i) &= \sigma^2, \quad i = 1, \ldots, n \\
\mathsf{Cov}(R_i, R_j) &= 0 \text{ for all } i \neq j.
\end{aligned}$$

Let w_i denote fraction of wealth invested in i^{th} security at time $t = 0$

- must have $\sum_{i=1}^{n} w_i = 1$ for any portfolio.

Consider now two portfolios:

Portfolio A: 100% invested in security 1 so that $w_1 = 1$ and $w_i = 0$ for $i > 1$.

Portfolio B: An equi-weighted portfolio so that $w_i = 1/n$ for $i = 1, \ldots, n$.

Then have

$$\begin{aligned}
\mathsf{E}[R_A] &= \mathsf{E}[R_B] = \mu \\
\mathsf{Var}(R_A) &= \sigma^2 \\
\mathsf{Var}(R_B) &= \sigma^2/n.
\end{aligned}$$

where R_A and R_B are random returns of portfolios A and B, respectively.

A Simple Motivating Example

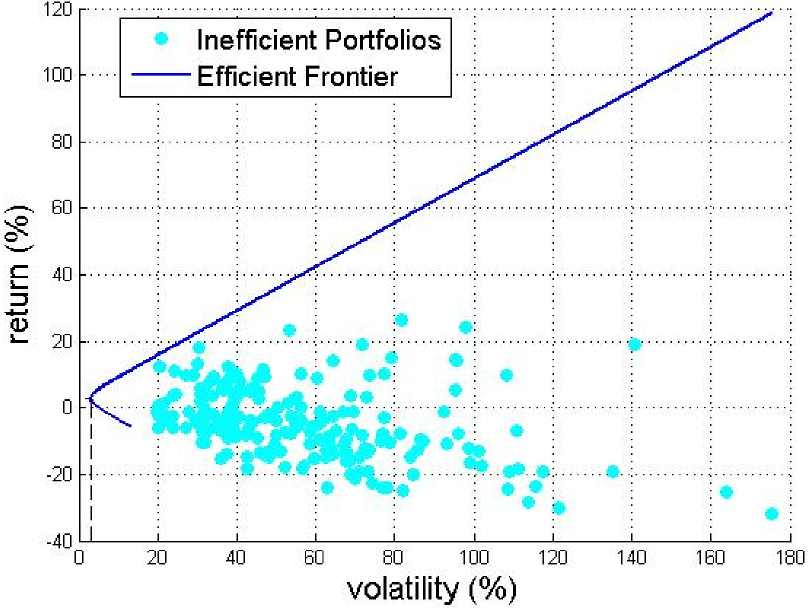
Both portfolios therefore have same expected return but very different return variances.

A risk-averse investor should clearly prefer portfolio B because this portfolio benefits from diversification without sacrificing any expected return.

- the central insight of Markowitz.

Consider figure on next slide:

- We simulated $m = 200$ random portfolios from universe of $n = 8$ securities.
- Expected return and volatility, i.e. standard deviation, plotted for each one
  - they are inefficient because each one can be improved.
- In particular, for same expected return it is possible to find an (efficient) portfolio with a smaller volatility.
- Alternatively, for same volatility it is possible to find an (efficient) portfolio with higher expected return.



Inefficient Portfolios
Efficient Frontier
return (%)
volatility (%)
120
100
80
60
40
20
0
-20
-40
0
20
40
60
80
100
120
140
160
180

Foundations of Financial Engineering

Mean-Variance without a Riskfree Asset

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Mean-Variance without a Riskfree Asset

Have n risky securities with corresponding return vector, $\mathbf{R}$, satisfying

$$\mathbf{R} \sim \text{MVN}_n(\mu, \Sigma).$$

Mean-variance portfolio optimization problem is formulated as:

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}^\top \Sigma w \tag{1}$$

$$\text{subject to} \quad \mathbf{w}^\top \mu = p$$
$$\text{and} \quad \mathbf{w}^\top \mathbf{1} = 1.$$

(1) is a quadratic program (QP)

- can be solved via standard Lagrange multiplier methods.

Note that specific value of p will depend on investor's level of risk aversion.

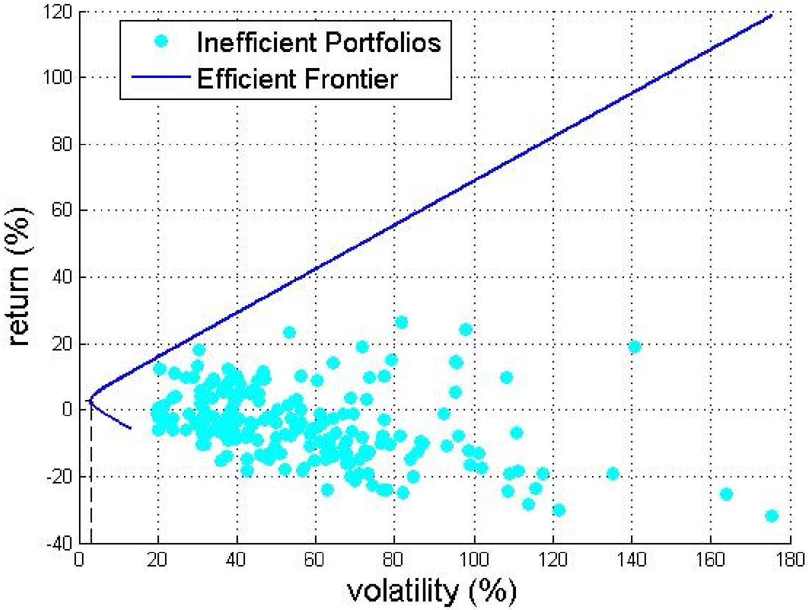
Mean-Variance without a Riskfree Asset

- When we plot the mean portfolio return, p, against the corresponding minimized portfolio volatility / standard deviation we obtain the so-called portfolio frontier.

- Can also identify the portfolio having minimal variance among all risky portfolios: the minimum variance portfolio.

  Let $\bar{R}_{mv}$ denote expected return of minimum variance portfolio.

- Points on portfolio frontier with expected returns greater than $\bar{R}_{mv}$ are said to lie on the efficient frontier.


Inefficient Portfolios
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return (%)
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180

A 2-Fund Theorem

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be mean-variance efficient portfolios corresponding to expected returns p_1 and p_2, respectively, with $p_1 \neq p_2$.

Can then be shown that all efficient portfolios can be obtained as linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$

- an example of a 2-fund theorem.

Question: How might you prove this?

Foundations of Financial Engineering

Mean-Variance with a Riskfree Asset

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Mean-Variance with a Riskfree Asset

Assume that there is a risk-free security available with risk-free rate equal to r_f.

Let $\mathbf{w} := (w_1, \ldots, w_n)^\top$ be the vector of portfolio weights on the n risky assets

- so $1 - \sum_{i=1}^{n} w_i$ is the weight on the risk-free security.

Investor's portfolio optimization problem may then be formulated as

$$\min_{\mathbf{w}} \ \frac{1}{2}\mathbf{w}^\top \Sigma \mathbf{w} \tag{2}$$

$$\text{subject to} \qquad \left(1 - \sum_{i=1}^{n} w_i\right) r_f \ + \ \mathbf{w}^\top \mu \ = \ p.$$

Mean-Variance with a Riskfree Asset

Optimal solution to (2) given by

$$\mathbf{w} \;=\; \xi\, \Sigma^{-1}(\mu - r_f \mathbf{1}) \tag{3}$$

where $\xi := \sigma^2_{min}/(p - r_f)$ and

$$\sigma^2_{min} \;=\; \frac{(p - r_f)^2}{(\mu - r_f\mathbf{1})^\top\, \Sigma^{-1}\, (\mu - r_f\mathbf{1})} \tag{4}$$

is the minimized variance.

While ξ (or p) depends on investor's level of risk aversion it is often inferred from the market portfolio.

Mean-Variance with a Riskfree Asset

Taking square roots across (4) we obtain

$$\sigma_{min}(p) \;=\; \frac{(p - r_f)}{\sqrt{(\mu - r_f\mathbf{1})^\top \, \Sigma^{-1} \, (\mu - r_f\mathbf{1})}} \tag{5}$$

– so the efficient frontier $(\sigma_{min}(p),\, p)$ is linear when we have a risk-free security:

Does the Frontier of Risky Assets (Only) Play Any Role?

Can gain further insight as follows:

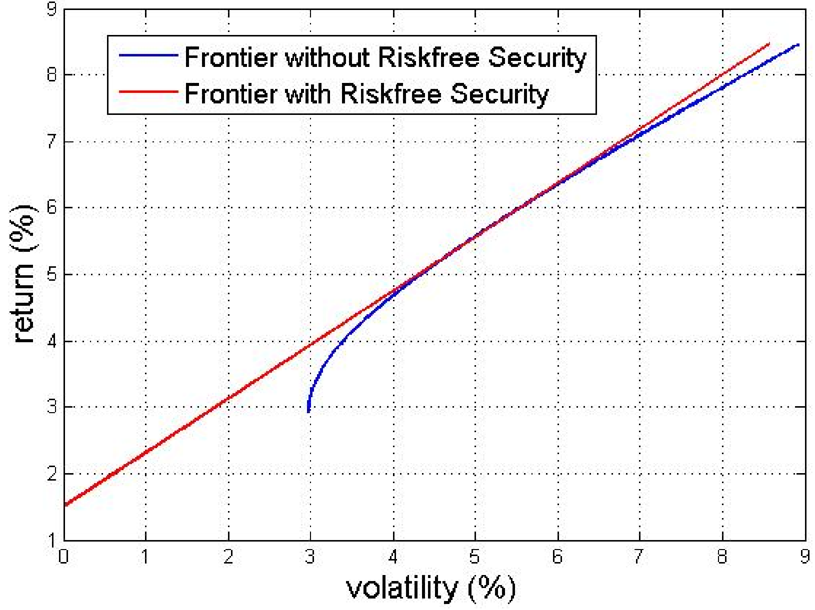
- Let R denote the (random) return of any portfolio of risky (only) securities.
- Now consider forming a portfolio of the risk-free security with this risky portfolio.
- Return on this new portfolio is

$$R_\alpha := \alpha r_f + (1-\alpha)R$$

- Also have

$$\begin{aligned}\bar{R}_\alpha &= \alpha r_f + (1-\alpha)\bar{R} \\ \sigma_\alpha &= (1-\alpha)\sigma_R\end{aligned}$$

So the mean and standard deviation of the portfolio varies linearly with α.


Frontier without Riskfree Security
Frontier with Riskfree Security
return (%)
volatility (%)
0
1
2
3
4
5
6
7
8
9

Mean-Variance with a Riskfree Asset

In fact suppose $r_f < \bar{R}_{mv}$.

Efficient frontier then becomes a straight line that is tangent to the risky efficient frontier and with a y-intercept equal to the risk-free rate.

We also then have a 1-fund theorem:

> Every investor will optimally choose to invest in a combination of the risk-free security and the tangency portfolio.

Mean-Variance with a Riskfree Asset

Recall the optimal solution to mean-variance problem given by:

$$\mathbf{w} \;=\; \xi\,\Sigma^{-1}(\mu - r_f\mathbf{1}) \tag{6}$$

where $\xi := \sigma^2_{min}/(p - r_f)$ and

$$\sigma^2_{min} \;=\; \frac{(p - r_f)^2}{(\mu - r_f\mathbf{1})^\top\,\Sigma^{-1}\,(\mu - r_f\mathbf{1})} \tag{7}$$

is the minimized variance.

The tangency portfolio, $\mathbf{w}^*$, is given by (6) except that it must be scaled so that its component sum to 1

- this scaling removes the dependency on p.

Question: Describe the efficient frontier if no-borrowing is allowed.

Mean-Variance with a Riskfree Asset

Definition: The Sharpe ratio of a portfolio (or security) is the ratio of the expected excess return of the portfolio to the portfolio's volatility.

Definition: The Sharpe optimal portfolio is the portfolio with maximum Sharpe ratio.

Have already seen(!) that the tangency portfolio, $\mathbf{w}^*$, is the Sharpe optimal portfolio.

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Weaknesses of Traditional Mean-Variance Analysis

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Weaknesses of Traditional Mean-Variance Analysis

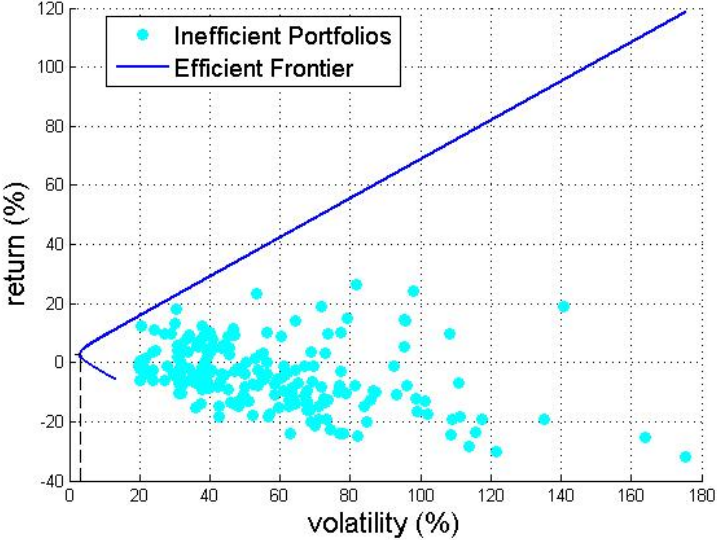
Traditional mean-variance analysis has many weaknesses when applied naively in practice.

For example, it often produces extreme portfolios combining extreme shorts with extreme longs

- portfolio managers generally do not trust these extreme weights as a result.

This problem is typically caused by estimation errors in the mean return vector and covariance matrix.

Consider again the same efficient frontier of risky securities that we saw earlier:



In practice, investors can never compute this frontier since they do not know the true mean vector and covariance matrix of returns.

The best we can hope to do is to approximate it. But how might we do this?

Weaknesses of Traditional Mean-Variance Analysis

One approach would be to simply estimate the mean vector and covariance matrix using historical data.

Each of dashed yellow curves in next figure is an estimated frontier that we computed by:

(i) simulating $m = 24$ sample returns from the true distribution
  - which, in this case, was assumed to be multivariate normal.

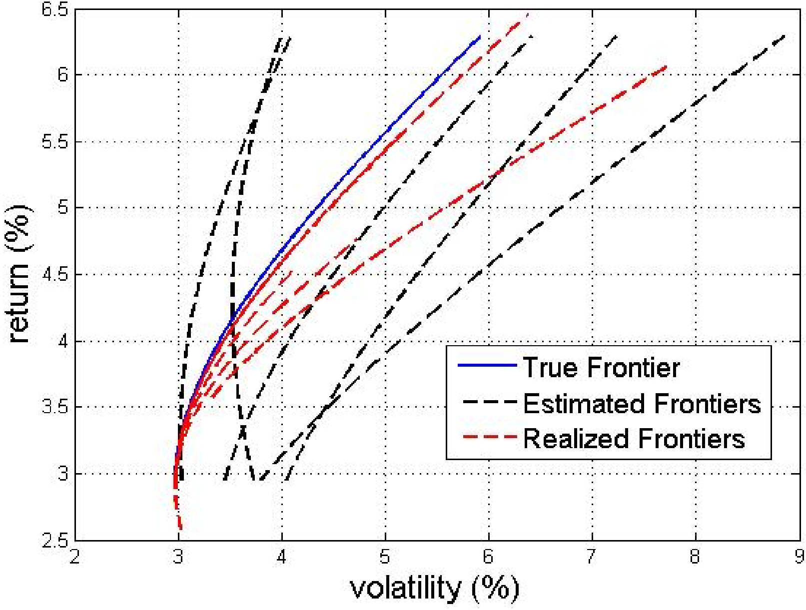
(ii) estimating the mean vector and covariance matrix from this simulated data

(iii) using these estimates to generate the (estimated) frontier.

The blue curve in the figure is the true frontier computed using the true mean vector and covariance matrix.

First observation is that the estimated frontiers are random and can differ greatly from the true frontier;

- an estimated frontier may lie below, above or may intersect the true frontier.



6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
3
4
5
6
7
8
9
return (%)
volatility (%)
True Frontier
Estimated Frontiers
Realized Frontiers

Weaknesses of Traditional Mean-Variance Analysis

An investor who uses such an estimated frontier to make investment decisions may end up choosing a poor portfolio.

But just how poor?

The dashed red curves in the figure are the realized frontiers

- the true mean - volatility tradeoff that results from making decisions based on the estimated frontiers.

In contrast to the estimated frontiers, the realized frontiers must always (why?) lie below the true frontier.

In the figure some of the realized frontiers lie very close to the true frontier and so in these cases an investor would do very well.

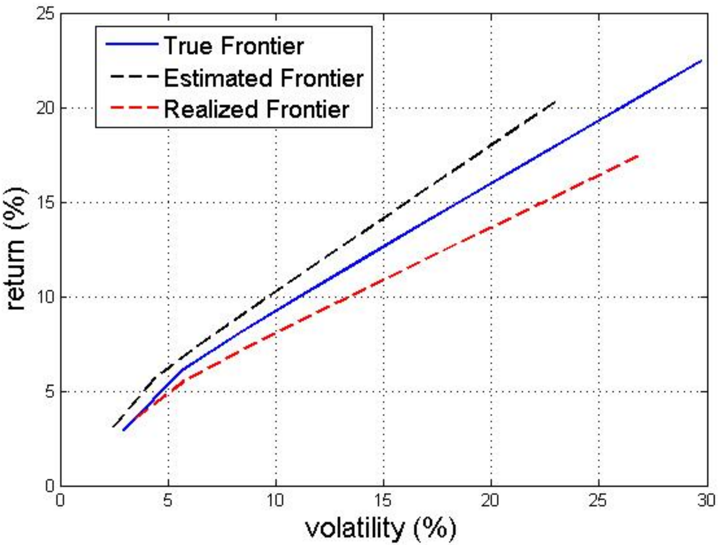
But in other cases the realized frontier is far from the (unobtainable) true efficient frontier.

Weaknesses of Traditional Mean-Variance Analysis

As a result of these weaknesses, portfolio managers traditionally had little confidence in mean-variance analysis and therefore applied it rarely in practice.

Efforts to overcome these problems include the use of better estimation techniques such as the use of shrinkage estimators, robust estimators and Bayesian techniques such as the Black-Litterman framework.

In addition to mitigating the problem of extreme portfolios, the Black-Litterman framework allows users to specify their own subjective views on the market in a consistent and tractable manner.



– Figure displays a robustly estimated frontier.

It lies much closer to the true frontier

- this is also the case with the corresponding realized frontier.

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The Capital Asset Pricing Model (CAPM)

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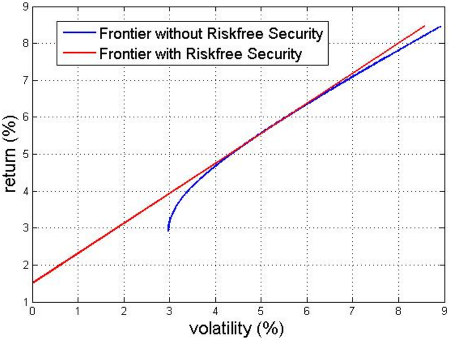
The Capital Asset Pricing Model (CAPM)



If every investor is a mean-variance optimizer then each of them will hold the same tangency portfolio of risky securities in conjunction with a position in the risk-free asset.

Because the tangency portfolio is held by all investors and because markets must clear, we can identify this portfolio as the market portfolio.

The efficient frontier is then termed the capital market line (CML).

The Capital Asset Pricing Model (CAPM)

Now let R_m and $\bar{R}_m$ denote the return and expected return, respectively, of the market, i.e. tangency, portfolio.

Central insight of the Capital Asset-Pricing Model is that in equilibrium the riskiness of an asset is not measured by the standard deviation of its return but by its beta:

$$\beta := \frac{\mathsf{Cov}(R, R_m)}{\mathsf{Var}(R_m)}.$$

In particular, there is a linear relationship between the expected return, $\bar{R} = \mathsf{E}[R]$, of any security (or portfolio) and the expected return of the market portfolio:

$$\bar{R} \;=\; r_f \;+\; \beta\,(\bar{R}_m - r_f). \tag{8}$$

The Capital Asset Pricing Model (CAPM)

In order to prove (8), consider a portfolio with weights α and weight $1-\alpha$ on the risky security and market portfolio, respectively.

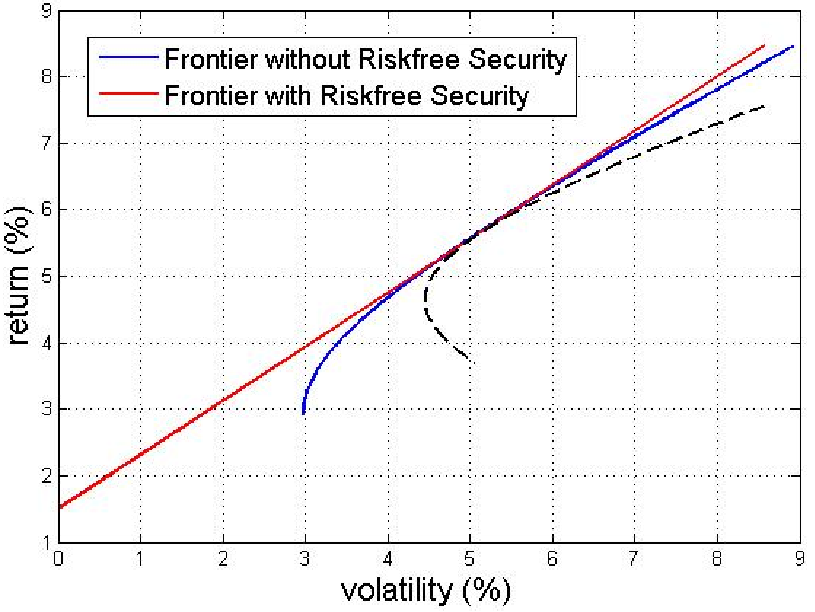
Let R_α denote the (random) return of this portfolio as a function of α.

Then have

$$\begin{aligned} \mathsf{E}[R_\alpha] &= \alpha\bar{R} + (1-\alpha)\bar{R}_m & (9) \\ \sigma^2_{R_\alpha} &= \alpha^2\sigma^2_R + (1-\alpha)^2\sigma^2_{R_m} + 2\alpha(1-\alpha)\sigma_{R,R_m}. & (10) \end{aligned}$$

As α varies, the mean and standard deviation, $(\mathsf{E}[R_\alpha], \sigma^2_{R_\alpha})$, trace out a curve.

But this curve cannot (why?) cross the efficient frontier ...



Frontier without Riskfree Security
Frontier with Riskfree Security
return (%)
volatility (%)
0
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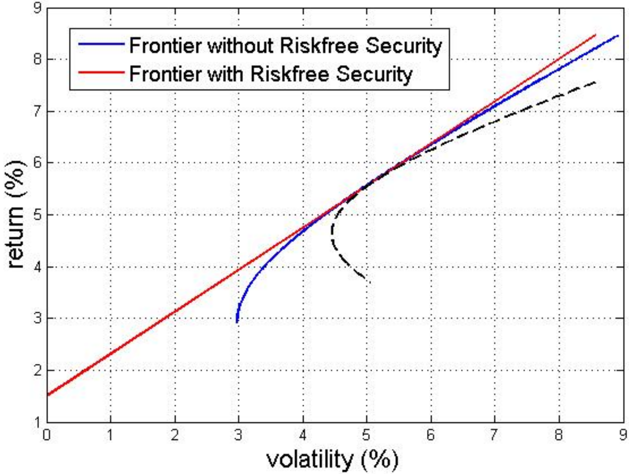
The Capital Asset Pricing Model (CAPM)

Therefore at $\alpha = 0$ this curve must be tangent to the CML.

So slope of the curve at $\alpha = 0$ must equal slope of the CML.

Using (9) and (10) we see slope of CML is given by

$$\begin{aligned} \left.\frac{d\,\mathsf{E}[R_\alpha]}{d\,\sigma_{R_\alpha}}\right|_{\alpha=0} &= \left.\frac{d\,\mathsf{E}[R_\alpha]}{d\,\alpha} \Big/ \frac{d\,\sigma_{R_\alpha}}{d\,\alpha}\right|_{\alpha=0} \\ &= \left.\frac{\sigma_{R_\alpha}\left(\bar{R} - \bar{R}_m\right)}{\alpha\sigma_R^2 - (1-\alpha)\sigma_{R_m}^2 + (1-2\alpha)\sigma_{R,R_m}}\right|_{\alpha=0} \\ &= \frac{\sigma_{R_m}\left(\bar{R} - \bar{R}_m\right)}{-\sigma_{R_m}^2 + \sigma_{R,R_m}}. \end{aligned}$$



Slope of CML is $\left(\bar{R}_m - r_f\right)/\sigma_{R_m}$ and equating the two therefore yields

$$\frac{\sigma_{R_m}\left(\bar{R} - \bar{R}_m\right)}{-\sigma_{R_m}^2 + \sigma_{R,R_m}} = \frac{\bar{R}_m - r_f}{\sigma_{R_m}} \tag{11}$$

which upon simplification gives (8).

The Capital Asset Pricing Model (CAPM)

The CAPM is one of the most famous models in all of finance.

Even though it arises from a simple one-period model, it provides considerable insight to the problem of asset-pricing.

For example, it is well-known that riskier securities should have higher expected returns in order to compensate investors for holding them. But how do we measure risk?

According to the CAPM, security risk is measured by its beta which is proportional to its covariance with the market portfolio

- a very important insight.

This does not contradict the mean-variance formulation of Markowitz where investors do care about return variance

- indeed, we derived the CAPM from mean-variance analysis!