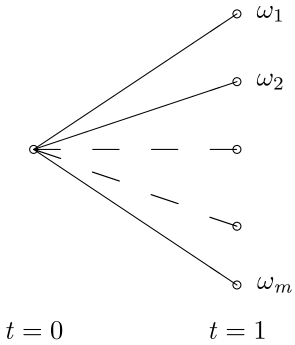
Foundations of Financial Engineering

Single Period Models

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Single Period Models



- There are $N + 1$ securities available for trading.
- At $t = 1$ one of m possible states will occur.
- $S_0^{(i)}$:= time 0 value of ith security, $0 \le i \le N$.
- $S_1^{(i)}(\omega_j)$:= its value at date $t = 1$ if outcome ω_j occurs.
- $\mathbb{P} = (p_1, \ldots, p_m)$:= the true probability distribution
  - vital assumption: $p_k > 0$ for each k.

Arbitrage in Single Period Models

Definition: A type A arbitrage is an investment that produces immediate positive reward at $t = 0$ and has no future cost at $t = 1$.

Definition: A type B arbitrage is an investment that has non-positive cost at $t = 0$ but has positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then.

We always assume that arbitrage opportunities do not exist.

Definition: Let $S_0^{(1)}$ and $S_0^{(2)}$ be the date $t = 0$ prices of two securities whose payoffs at date $t = 1$ are $d_1 \in \mathbb{R}^m$ and $d_2 \in \mathbb{R}^m$, respectively. We say that linear pricing holds if for all α_1 and α_2, $\alpha_1 S_0^{(1)} + \alpha_2 S_0^{(2)}$ is the value of the security that pays $\alpha_1 d_1 + \alpha_2 d_2$ at date $t = 1$.

Easy to see absence of type A arbitrage $\Rightarrow$ linear pricing holds. Why?

Always assume no arbitrage opportunities $\Rightarrow$ always assume linear pricing holds

Elementary Securities, Attainability and State Prices

Definition: An elementary security is a security with date $t = 1$ payoff of the form $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. a payoff of 1 in state j and 0 otherwise.

As there are m possible states at $t = 1$, there are m elementary securities.

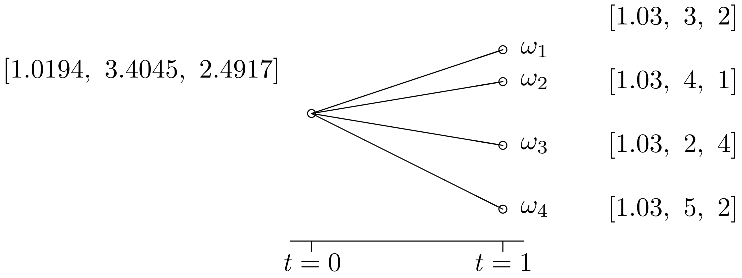
Definition: A security or *contingent claim*, X, is attainable if there exists a trading strategy, $\theta = [\theta_0 \; \theta_1 \; \ldots \; \theta_N]^\top$, such that

$$\begin{bmatrix} X(\omega_1) \\ \vdots \\ X(\omega_m) \end{bmatrix} = \begin{bmatrix} S_1^{(0)}(\omega_1) & \ldots & S_1^{(N)}(\omega_1) \\ \vdots & \vdots & \vdots \\ S_1^{(0)}(\omega_m) & \ldots & S_1^{(N)}(\omega_m) \end{bmatrix} \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_N \end{bmatrix}. \qquad (1)$$

In shorthand, $X = S_1 \theta$ and we call θ the replicating portfolio for X.

An Attainable Claim

Example 1: $m = 4$ possible states of nature and 3 securities, i.e. $N = 2$.



The claim $X = [7.47 \;\; 6.97 \;\; 9.97 \;\; 10.47]^\top$ is attainable. Why?

Because $X = S_1\theta$ where $\theta = [-1 \;\; 1.5 \;\; 2]^\top$

- so θ is a replicating portfolio for X.

Note that attainability does not depend on date $t = 0$ cost of the securities.

Foundations of Financial Engineering

Arbitrage, State Prices and Numeraire Securities

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

A More Formal Definition of Arbitrage

Definition: A type A arbitrage is a trading strategy, θ, such that $S_0^\top \theta < 0$ and $S_1 \theta = 0$.

Definition: A type B arbitrage is a trading strategy, θ, such that $S_0^\top \theta \leq 0$, $S_1 \theta \geq 0$ and $S_1 \theta \neq 0$.

Note if $S_0^\top \theta < 0$ then θ has negative cost and so produces an immediate positive reward if purchased at $t = 0$.

Definition: Say that a vector $\pi = [\pi_1 \ \ldots \ \pi_m]^\top > 0$ is a vector of state prices if the date $t = 0$ price, P, of any attainable security, X, satisfies

$$P = \sum_{k=1}^{m} \pi_k X(\omega_k). \tag{2}$$

We call π_k the k^{th} state price.

State Prices

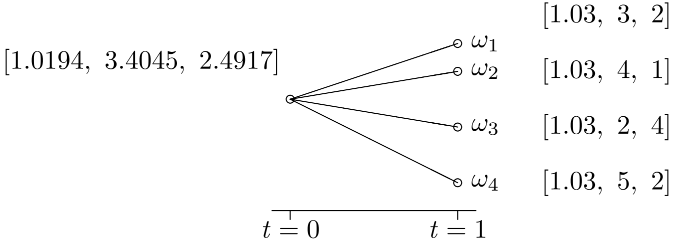
Remark: There might be many state price vectors. If k^{th} elementary security is attainable, then it's price must be π_k and so k^{th} component of all possible state price vectors must therefore coincide.

Can easily check

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4]^\top = [0.2433\ \ 0.1156\ \ 0.3140\ \ 0.3168]^\top$$

is a vector of state prices in model of Example 1 below.

Example 1 ctd:



State Prices

More generally, however,

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3102 \\ 0.4113 \\ 0.2682 \end{bmatrix} + \epsilon \begin{bmatrix} 0.7372 \\ -0.5898 \\ -0.2949 \\ 0.1474 \end{bmatrix}$$

is also a vector of state prices for any ϵ such that $\pi_i > 0$ for $1 \leq i \leq 4$.

Numeraire Securities

Definition: A numeraire security is a security with a strictly positive price at all times, t.

Often convenient to express a security price in units of a chosen numeraire. For example, if the nth security is the numeraire security, then define

$$\overline{S}_t^{(i)}(\omega_j) \; := \; \frac{S_t^{(i)}(\omega_j)}{S_t^{(n)}(\omega_j)}$$

to be the date t, state ω_j price (in units of the numeraire security) of the i^{th} security. We say that we are deflating by the nth or numeraire security.

Remark: Deflated price of the numeraire security is always equal to 1.

Definition: The cash account is a security that earns interest at the risk-free rate of interest. In a single period model, the date $t = 1$ value of the cash account is $(1 + r) \times$ initial price, regardless of terminal state.

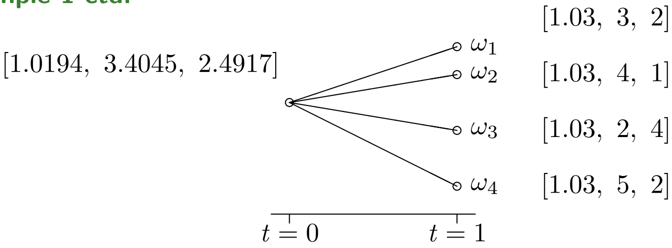
Numeraire Securities

- In practice, often deflate by the cash account if it is available.
- Note that deflating by the cash account $\equiv$ the usual process of discounting.
- Will use the zeroth security with price process, $S_t^{(0)}$, to denote the cash account whenever it is available.

Note that any security in Example 1 could serve as a numeraire security since each of the 3 securities has a strictly positive price process.

Also clear that the 0^{th} security there is actually the cash account.

Example 1 ctd:



Foundations of Financial Engineering

Equivalent Martingale Measures and (In)Complete Markets

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Equivalent Martingale Measure (EMM)

Let $S_t^{(n)}$ denote the date t price of our chosen numeraire security.

Definition: An equivalent martingale measure (EMM) or risk-neutral probability measure is a set of probabilities, $\mathbb{Q} = (q_1, \ldots, q_m)$ such that

1. $q_k > 0$ for all k.
2. The deflated security prices are martingales. That is

$$\overline{S}_0^{(i)} \;:=\; \frac{S_0^{(i)}}{S_0^{(n)}} \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_1^{(i)}}{S_1^{(n)}}\right] \;=:\; \mathsf{E}_0^{\mathbb{Q}}\left[\overline{S}_1^{(i)}\right]$$

   for all i where $\mathsf{E}_0^{\mathbb{Q}}[.]$ denotes expectation with respect to the risk-neutral probability measure, $\mathbb{Q}$.

Remark: Note that the EMM is specific to the chosen numeraire security, $S_t^{(n)}$. In fact it would be more accurate to speak of an EMM-numeraire pair.

Complete Markets

Now assume there are no arbitrage opportunities. If there is a full set of m elementary securities available, i.e. they are all attainable, then can use the state prices to compute the date $t = 0$ price, P, of any security.

To see this, let $x = [x_1 \; \ldots \; x_m]^\top$ be the vector of possible date $t = 1$ payoffs of a particular security. May then write

$$x = \sum_{i=1}^{m} x_i e_i$$

and use linear pricing to obtain $P = \sum_{i=1}^{m} x_i \pi_i$.

So if a full set of elementary securities exists, then we can construct and price every possible security.

Definition: If every random variable X is attainable, then market is complete. Otherwise we have an incomplete market.

Martingale Pricing Theory: Single-Period Models

Proposition 1: There are no arbitrage opportunities if an EMM, $\mathbb{Q}$, exists.

Proof: First recall that a type A arbitrage is a trading strategy, θ, such that $S_0^\top \theta < 0$ and $S_1 \theta = 0$.

Also recall that $\mathbb{Q} = (q_1, \ldots, q_m) > 0$ is an EMM if for all i

$$\overline{S}_0^{(i)} \;:=\; \frac{S_0^{(i)}}{S_0^{(n)}} \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_1^{(i)}}{S_1^{(n)}}\right] \;=:\; \mathsf{E}_0^{\mathbb{Q}}\left[\overline{S}_1^{(i)}\right].$$

Remark: If we did not insist that $q_k > 0$ in the definition of an EMM then the proposition would not hold. Why?

Martingale Pricing Theory: Single-Period Models

Theorem 2: Assume there is a security with strictly positive price process, $S_t^{(n)}$.

1. If there is a set of positive state prices, then a risk-neutral probability measure, $\mathbb{Q}$, exists with $S_t^{(n)}$ as the numeraire security.
2. There is a one-to-one correspondence between sets of positive state prices and risk-neutral probability measures for the given numeraire.

Martingale Pricing Theory: Single-Period Models

Proof: Suppose a set of positive state prices, $\pi = [\pi_1 \ \ldots \ \pi_m]^\top$, exists. For all j we then have

$$
\begin{aligned}
S_0^{(j)} &= \sum_{k=1}^{m} \pi_k S_1^{(j)}(\omega_k) \\
&= \left(\sum_{l=1}^{m} \pi_l S_1^{(n)}(\omega_l) \right) \sum_{k=1}^{m} \frac{\pi_k S_1^{(n)}(\omega_k)}{\sum_{l=1}^{m} \pi_l S_1^{(n)}(\omega_l)} \, \frac{S_1^{(j)}(\omega_k)}{S_1^{(n)}(\omega_k)}. \qquad (3)
\end{aligned}
$$

If we define

$$
q_k := \frac{\pi_k S_1^{(n)}(\omega_k)}{\sum_{l=1}^{m} \pi_l S_1^{(n)}(\omega_l)}, \qquad (4)
$$

then $\mathbb{Q} := (q_1, \ldots, q_m)$ defines a probability measure.

Also observe $\sum_{l=1}^{m} \pi_l S_1^{(n)}(\omega_l) = S_0^{(n)}$.

(3) and (4) then imply

$$\frac{S_0^{(j)}}{S_0^{(n)}} = \sum_{k=1}^{m} q_k \frac{S_1^{(j)}(\omega_k)}{S_1^{(n)}(\omega_k)} = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_1^{(j)}}{S_1^{(n)}}\right] \tag{5}$$

and so $\mathbb{Q}$ is an EMM, as desired.

The one-to-one correspondence between sets of positive state prices and risk-neutral probability measures is clear from (4)

- the denominator on rhs of (4) is just a normalizing constant. $\square$

Remark. The true real-world probabilities, $\mathbb{P} = (p_1, \ldots, p_m)$, are almost irrelevant here. The only connection between $\mathbb{P}$ and $\mathbb{Q}$ is that they must be equivalent, i.e. $p_k > 0 \Leftrightarrow q_k > 0$.

Foundations of Financial Engineering

The Fundamental Theorems of Asset Pricing

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

No Arb. $\equiv$ Existence of Pos. State Prices $\equiv$ Existence of EMM

Theorem 3: In a one-period model there is no arbitrage if and only if there exists a set of positive state prices.

Proof: (i) Suppose there exists a set of positive state prices. If there also exists a numeraire security then by Prop 1. and Theorem 2 there is no arbitrage.

(If a numeraire security does not exist then we can show directly that there cannot be an arbitrage.)

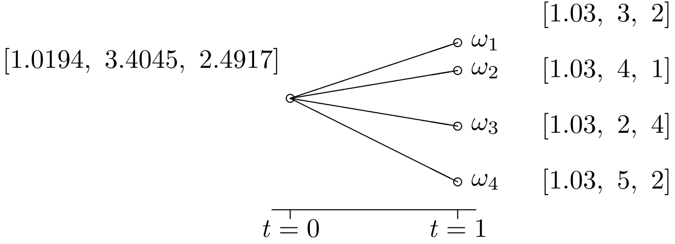
(ii) Other direction relies on linear programming duality.

No Arb. ≡ Existence of Pos. State Prices ≡ Existence of EMM

We now have:

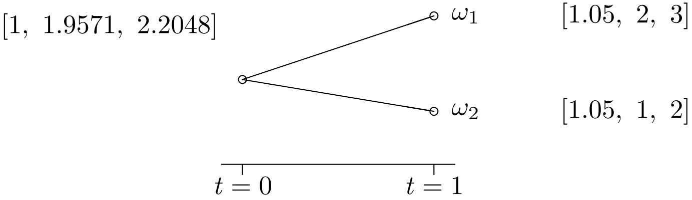
Theorem 4: (First Fundamental Theorem of Asset Pricing)
Assume there exists a security with strictly positive price process. Then the absence of arbitrage, the existence of state prices and the existence of an EMM, $\mathbb{Q}$, are all equivalent.

Example 1 ctd: This model is arbitrage-free since we saw earlier that a vector of positive state prices exists.



A Market with Arbitrage Opportunities

Example 2: Consider the following one-period, 2-state model:



No positive state price vector exists for this model so there must be an arbitrage opportunity.

Exercise: Find an arbitrage strategy, θ, in this model.

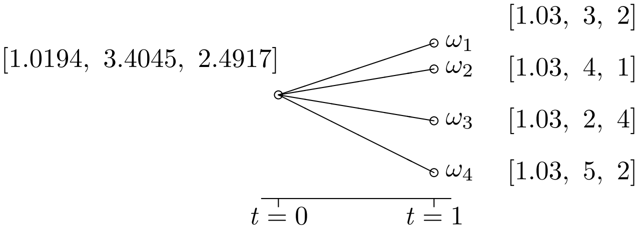
Complete / Incomplete Markets

Theorem 5: Assume there are no arbitrage opportunities. Then the market is complete if and only if the matrix of date $t = 1$ payoffs, S_1, has rank m.

Proof: The proof is immediate!

An Incomplete Market

Example 1 ctd:

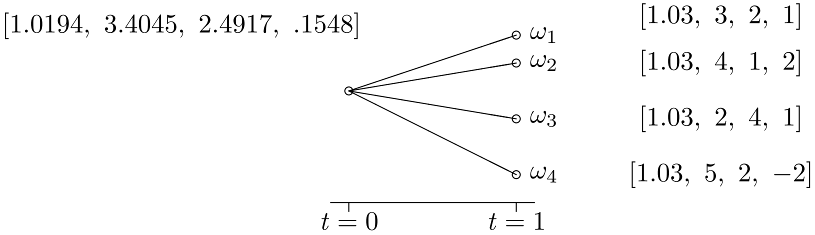


The model is arbitrage-free since we saw earlier that a vector of positive state prices exists.

But the model is incomplete since rank of payoff matrix, S_1, can be at most 3 which is less than the number of possible states, 4.

A Complete Market

Example 3: Consider one-period model below where there are $m = 4$ possible states of nature and 4 securities, i.e $N = 3$.



Can easily check that rank$(S_1) = 4 = m$, so this model is indeed complete.

Can also confirm model is arbitrage-free by checking

$$[\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]^\top := [0.2433 \ \ 0.1156 \ \ 0.3140 \ \ 0.3168]^\top$$

is a (unique) state price vector.

Exercise: Show that if a market is incomplete, then at least one elementary security is not attainable.

The Second Fundamental Theorem of Asset Pricing

Theorem 6: (Second Fundamental Theorem of Asset Pricing)
Assume there exists a security with strictly positive price process and there are no arbitrage opportunities.

Then the market is complete if and only if there exists exactly one equivalent martingale measure or equivalently, one vector of positive state prices.

Proof: **(i)** Suppose first that market is complete. Then there exists a unique set of positive state prices, and therefore by Theorem 2 a unique risk-neutral probability measure.

(i) Other direction relies on linear programming duality theory.

Foundations of Financial Engineering

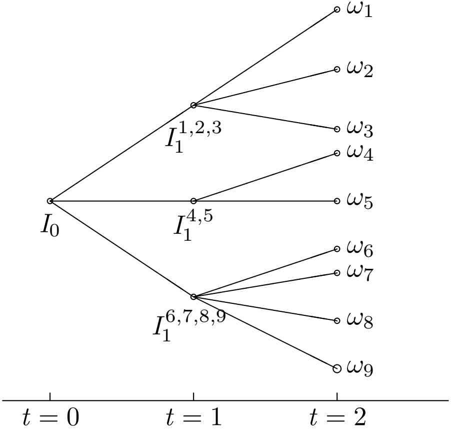
Multi-Period Models

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Multi-Period Models

Example



First need to extend some single-period definitions and introduce concepts of trading strategies and self-financing trading strategies.

Will assume no-dividends for now – but the extension to dividends is easy.

Multi-Period Models

- As before there are $N+1$ securities, m possible states of nature and that true probability measure is denoted by $\mathbb{P} = (p_1, \ldots, p_m) > 0$.

- The investment horizon is $[0, T]$ and there are a total of T trading periods.

- Securities may therefore be purchased or sold at any date t for $t = 0, 1, \ldots, T-1$.

- Our example shows a typical multi-period model with $T = 2$ and $m = 9$ possible states.

- The manner in which information is revealed as time elapses is clear from this model.

  e.g. At node $I_1^{4,5}$ the available information tells us that the true state of the world is either ω_4 or ω_5.

Multi-Period Models

- Multi-period model is composed of a series of single-period models.

  e.g. At date $t = 0$ there is a single one-period model corresponding to node I_0. Similarly at date $t = 1$ there are three possible one-period models corresponding to nodes $I_1^{1,2,3}$, $I_1^{4,5}$ and $I_1^{6,7,8,9}$, respectively.

- The particular one-period model that prevails at $t = 1$ will depend on the true state of nature.

- Given a probability measure, $\mathbb{P} = (p_1, \ldots, p_m)$, can easily compute conditional probabilities of each state.

  e.g. $\mathbf{P}(\omega_1 | I_1^{1,2,3}) = p_1/(p_1 + p_2 + p_3)$.

  Conditional probabilities can be interpreted as probabilities in corresponding single-period models.

  e.g. $p_1 = \mathbf{P}(I_1^{1,2,3} | I_0) \ \mathbf{P}(\omega_1 | I_1^{1,2,3})$.

  This observation allows us to easily generalize single-period results to multi-period models.

Predictable Trading Strategies

Definition: A predictable stochastic process is a process whose time t value, X_t, is known at time $t-1$ given all the information that is available at time $t-1$.

Definition: A trading strategy is a vector, $\theta_t = [\theta_t^{(0)}(\omega) \; \ldots \; \theta_t^{(N)}(\omega)]^\top$, of predictable stochastic processes that describes the number of units of each security held just before trading at time t, as a function of t and ω.

So $\theta_t^{(i)}(\omega) = \#$ of units of ith security held between times $t-1$ and t in state ω

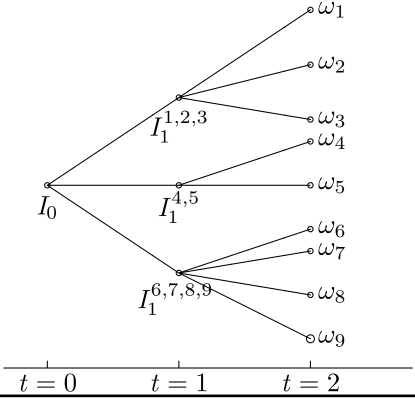
- if $\theta_t^{(i)}$ negative then it equals number of units sold short.

Important to emphasize that θ_t is known at date $t-1$.

Constraints Imposed by Predictability

Example: Referring to earlier 2-period model, must be the case that:

$$
\begin{aligned}
\theta_2^{(i)}(\omega_1) &= \theta_2^{(i)}(\omega_2) = \theta_2^{(i)}(\omega_3) \\
\theta_2^{(i)}(\omega_4) &= \theta_2^{(i)}(\omega_5) \\
\theta_2^{(i)}(\omega_6) &= \theta_2^{(i)}(\omega_7) = \theta_2^{(i)}(\omega_8) = \theta_2^{(i)}(\omega_9).
\end{aligned}
$$



Foundations of Financial Engineering

Self-Financing Trading Strategies, Complete Markets and EMM's

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Self-Financing Trading Strategies

Definition: The value process, $V_t(\theta)$, associated with a trading strategy, θ_t, is defined by

$$V_t = \begin{cases} \sum_{i=0}^{N} \theta_1^{(i)} S_0^{(i)} & \text{for } t = 0 \\ \sum_{i=0}^{N} \theta_t^{(i)} S_t^{(i)} & \text{for } t \geq 1. \end{cases}$$

Definition: A self-financing trading strategy is a strategy, θ_t, where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds.

In particular, a self-financing strategy satisfies

$$V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)} S_t^{(i)} \quad \text{for } t = 1, \ldots, T-1.$$

Self-Financing Trading Strategies

Exercise: Show that if a trading strategy, θ_t, is self-financing then the corresponding value process, V_t, satisfies

$$V_{t+1} - V_t = \sum_{i=0}^{N} \theta_{t+1}^{(i)} \left(S_{t+1}^{(i)} - S_t^{(i)} \right). \tag{6}$$

Clearly then changes in the value of the portfolio are due to capital gains or losses and are not due to the injection or withdrawal of funds.

Can also write (6) as $dV_t = \theta_t^\top \, dS_t$

- anticipates continuous-time definition of self-financing.

Arbitrage in Multi-Period Models

Can now define arbitrage in our multi-period setting:

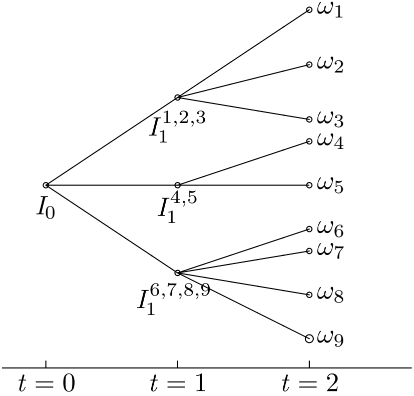
Definition: A type A arbitrage opportunity is a self-financing trading strategy, θ_t, such that $V_0(\theta) < 0$ and $V_T(\theta) = 0$.

Definition: A type B arbitrage opportunity is a self-financing trading strategy, θ_t, such that $V_0(\theta) = 0$, $V_T(\theta) \geq 0$ and $\mathsf{E}_0^{\mathbb{P}}[V_T(\theta)] > 0$.

Contingent Claims and Attainability

Definition: A contingent claim, C, is a random variable whose value at time T is known at that time given the information available then.

Definition: A contingent claim C is attainable if there exists a self-financing trading strategy, θ_t, whose value process, V_T, satisfies $V_T = C$.

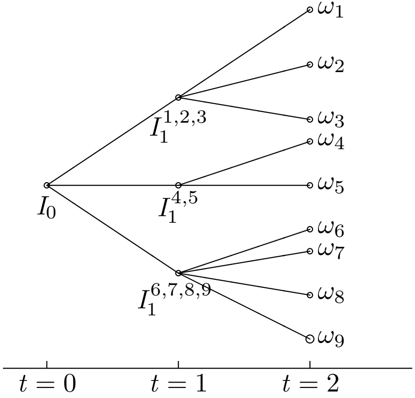


Complete and Incomplete Markets

Can now extend definition of completeness:

Definition: We say a market is complete if every contingent claim is attainable. Otherwise the market is incomplete.

Note that above definitions of attainability and (in)completeness are consistent with single-period definitions.



Equivalent Martingale Measures (EMMs)

Definitions of numeraire security and cash account are unchanged from one-period definitions.

Can now give multi-period definition of an equivalent martingale measure (EMM), or set of risk-neutral probabilities:

Assume again we have a specific numeraire security with price process, $S_t^{(n)}$.

Definition: An equivalent martingale measure (EMM), $\mathbb{Q} = (q_1, \ldots, q_m)$, is a set of probabilities such that

1. $q_i > 0$ for all $i = 1, \ldots, m$.
2. The deflated security prices are martingales. That is

$$\overline{S}_t^{(i)} := \frac{S_t^{(i)}}{S_t^{(n)}} = \mathsf{E}_t^{\mathbb{Q}}\left[\frac{S_{t+s}^{(i)}}{S_{t+s}^{(n)}}\right] =: \mathsf{E}_t^{\mathbb{Q}}\left[\overline{S}_{t+s}^{(i)}\right]$$

   for $s, t \geq 0$, for all $i = 0, \ldots, N$.

Foundations of Financial Engineering

The Fundamental Theorems in Multi-Period Models

Martin B. Haugh
Department of Industrial Engineering and Operations Research
Columbia University

Equivalent Martingale Measures (EMMs)

Proposition 7: If an EMM, $\mathbb{Q}$, exists, then the deflated value process, V_t, of any self-financing trading strategy is a $\mathbb{Q}$-martingale.

Proof. Let θ_t be the self-financing trading strategy and let $\overline{V}_{t+1} := V_{t+1}/S^{(n)}_{t+1}$ denote the deflated value process. We then have

$$\begin{aligned}
\mathsf{E}_t^{\mathbb{Q}}\left[\overline{V}_{t+1}\right] &= \mathsf{E}_t^{\mathbb{Q}}\left[\sum_{i=0}^{N} \theta_{t+1}^{(i)} \overline{S}_{t+1}^{(i)}\right] \\
&= \sum_{i=0}^{N} \theta_{t+1}^{(i)} \mathsf{E}_t^{\mathbb{Q}}\left[\overline{S}_{t+1}^{(i)}\right] \\
&= \sum_{i=0}^{N} \theta_{t+1}^{(i)} \overline{S}_{t}^{(i)} \\
&= \overline{V}_t
\end{aligned}$$

demonstrating that $\overline{V}_t$ is indeed a $\mathbb{Q}$-martingale, as claimed. □

Proposition 8: There can be no arbitrage opportunities if an EMM, Q, exists.

Absence of Arbitrage $\equiv$ Existence of EMM

Can now now state principal result for multi-period models (assuming a numeraire security exists):

Theorem 9: (First Fundamental Theorem of Asset Pricing)
In the multi-period model no arbitrage $\Leftrightarrow$ existence of an EMM, Q.

Outline Proof: **(i)** Suppose first there is no arbitrage.

Then can easily argue no arbitrage in any of the embedded one-period models.

This then implies that each embedded one-period model has a set of risk-neutral probabilities.

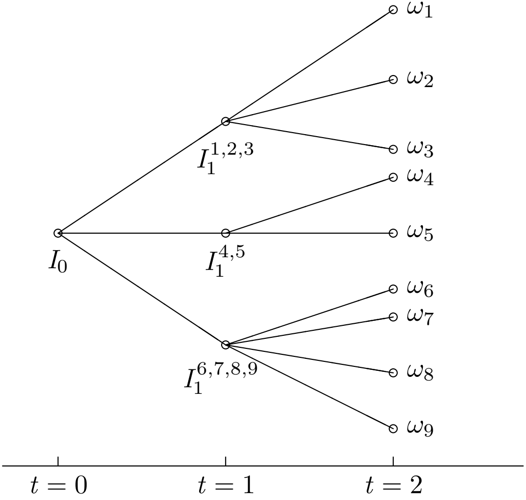
By multiplying these probabilities appropriately, can construct a multi-period EMM, Q.

(ii) Suppose there exists an EMM, Q. Then previous proposition gives result. $\square$

Complete Markets

Proposition 10: The market is complete if and only if every embedded one-period model is complete.

Exercise: Prove this proposition.



Complete Markets $\equiv$ Unique EMM

Theorem 11: (Second Fundamental Theorem of Asset Pricing)
Assume there exists a security with strictly positive price process and that there are no arbitrage opportunities.

Then the market is complete if and only if there exists exactly one risk-neutral martingale measure, $\mathbb{Q}$.

Outline Proof: (i) Suppose the market is complete. Then Prop. 10 implies every embedded one-period model is complete so every embedded one-period model has a unique EMM. Therefore the multi-period model has a unique EMM, $\mathbb{Q}$.

(ii) Suppose now $\mathbb{Q}$ is unique. Then the EMM corresponding to each one-period model is also unique. Therefore each one-period model is complete and so the multi-period model is complete. $\square$

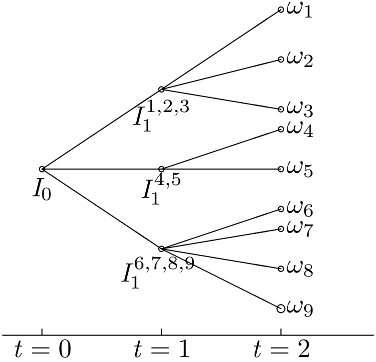
An Aside: State Prices

As in single-period models, also have an equivalence between EMMs, $\mathbb{Q}$, (for a given numeraire) and sets of state prices.

Use $\pi_t^{\{t+s\}}(\Lambda)$ to denote time t price of a security that pays $\$1$ at time $t+s$ in event that $\omega \in \Lambda$

- implicity assuming we can tell at time $t+s$ whether or not $\omega \in \Lambda$.

Example: $\pi_0^{\{1\}}(\{\omega_4, \omega_5\})$ is a valid expression whereas $\pi_0^{\{1\}}(\{\omega_4\})$ is not.



Foundations of Financial Engineering

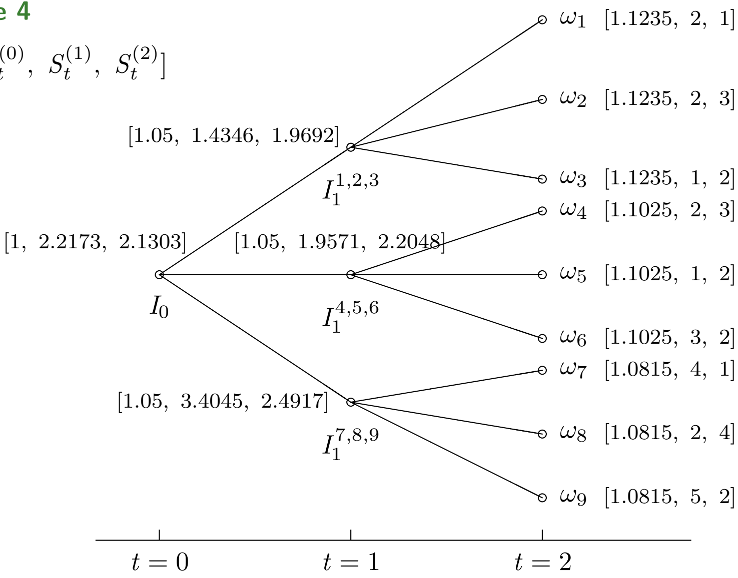
Some Multi-Period Examples

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Example 4

Key: $[S_t^{(0)},\ S_t^{(1)},\ S_t^{(2)}]$



- Is there a cash account in this market?
- Any immediate observations regarding market (in)completeness?

Further Questions & Exercises

1. Are there any arbitrage opportunities in this market?

2. If not, is this a complete or incomplete market?

3. Compute the state prices in this model.

4. Compute the risk-neutral, i.e. martingale probabilities, when we discount by the cash account, i.e., the 0^{th} security.

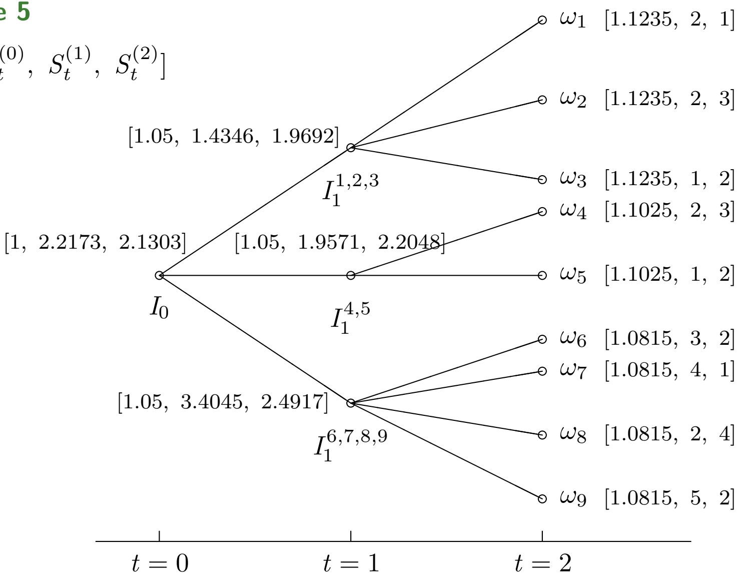
5. Compute the risk-neutral probabilities when we discount by the 2^{nd} security.

6. Using the state prices, find the price of a call option on the the 1^{st} asset with strike $k = 2$ and expiration date $t = 2$.

7. Confirm your answer to Q.6 by recomputing the option price using the EMM from Q.5.

Example 5

Key: $[S_t^{(0)},\ S_t^{(1)},\ S_t^{(2)}]$



- Is there a cash account in this market?
- Any immediate observations regarding market (in)completeness?

Further Questions & Exercises

1. Is this model arbitrage free?

2. Suppose the prices of the three securities were such that there were no arbitrage opportunities.

   Without bothering to compute such prices, do you think the model would then be a complete or incomplete model?

3. Suppose again that security prices were such that there were no arbitrage opportunities.

   Give a simple argument for why forward contracts are attainable

   - which means we can therefore price them in this model.

Foundations of Financial Engineering

Dividends and Intermediate Cash-Flows

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Dividends and Intermediate Cash-Flows

Consider a dividend-paying stock. No problem in one-period model since $t = 1$ dividend can simply be added to $t = 1$ value of the security.

In multi-period models, all results go through – as long as we are careful with our "bookkeeping".

For example, if θ_t is a self-financing strategy in a model with dividends then corresponding value process, V_t, should satisfy

$$V_{t+1} - V_t \;=\; \sum_{i=0}^{N} \theta_{t+1}^{(i)} \left(S_{t+1}^{(i)} + D_{t+1}^{(i)} - S_t^{(i)} \right). \tag{7}$$

Note that time t dividends, $D_t^{(i)}$, do not appear in (7) since S_t and V_t represent ex-dividend prices.

Must also redefine an EMM: now require the deflated cumulative gains process to be a $\mathbb{Q}$-martingale.

The cumulative gain process, G_t, of a security at time t = value of security at time t plus accumulated cash payments that result from holding the security.

Dividends and Intermediate Cash-Flows

Definitions of complete and incomplete markets are unchanged and the fundamental theorems still hold with new definition of EMM, self-financing strategies etc.

e.g. If model is arbitrage-free then there exists an EMM, $\mathbb{Q}$, such that

$$\overline{S}_t \;=\; \mathsf{E}_t^{\mathbb{Q}}\left[\sum_{j=t+1}^{t+s} \overline{D}_j \;+\; \overline{S}_{t+s}\right]$$

where $D_j =$ time j dividend that you receive if you hold one unit of the security, and S_t is its time t ex-dividend price.

If a security pays dividends then we cannot use it as the numeraire.

Instead can use the security's cumulative gains process as the numeraire

- as long as this gains process is strictly positive
- this makes sense as it is the gains process that represents the true value dynamics of holding the security.

Using a Dividend-Paying Security as the Numeraire

Easy (though tedious!) to check that all results regarding existence of EMMs and complete markets go through as before.

Just have to view each dividend as a separate security with S_t then interpreted as the price of the portfolio consisting of these 'dividend' securities as well as a security that is worth S_{t+s} at date $t+s$.

Alternatively, could imagine re-investing dividends back into the securities to obtain a non-dividend model.

Foundations of Financial Engineering

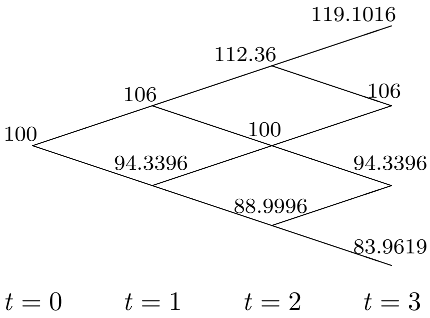
Martingale Pricing and the Binomial Model

Martin B. Haugh

Department of Industrial Engineering and Operations Research

Columbia University

Martingale Pricing and the Binomial Model



Cash account available and gross risk-free rate per period $= R$.

Stock goes up by a factor of u or down by factor of d in each period. The lattice therefore recombines:

- very advantageous for pricing non-path dependent derivatives
- all embedded one-period models are identical.

The Binomial Model

- No arbitrage if $d < R < u$. Why?

- Model is complete. Why?

A Counterintuitive Result: Pricing a European Call Option with $K = \$95$

Gross Risk-free Rate $= 1.02$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price

			24.10
		19.22	11.00
	14.76	7.08	0.00
11.04	4.56	0.00	0.00
t=0	t=1	t=2	t=3

Gross Risk-free Rate $= 1.04$

Stock Price

			119.10
		112.36	106.00
	106.00	100.00	94.34
100.00	94.34	89.00	83.96
t=0	t=1	t=2	t=3

European Option Price

			24.10
		21.01	11.00
	18.19	8.76	0.00
15.64	6.98	0.00	0.00
t=0	t=1	t=2	t=3

Question: Is this surprising?

Foundations of Financial Engineering

Martingale Pricing: Futures Contracts

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Calculating Futures Prices

Assume: (i) complete markets (ii) cash account is numeraire (iii) no dividends.

Let F_k = date k price of a futures contract written on a particular underlying security with process S_k.

Contract expires after n periods. Then know that $F_n = S_n$. Why?

Can compute time $t = n - 1$ futures price, F_{n-1}, by noting that (why?)

$$0 = \mathsf{E}_{n-1}^{\mathbb{Q}}\left[\frac{F_n - F_{n-1}}{B_n}\right].$$

Therefore obtain $F_{n-1} = \mathsf{E}_{n-1}^{\mathbb{Q}}[F_n]$.

Calculating Futures Prices

Same argument yields $F_k = \mathsf{E}_k^{\mathbb{Q}}[F_{k+1}]$ for $0 \leq k < n$.

Law of iterated expectations then implies $F_0 \; = \; \mathsf{E}_0^{\mathbb{Q}}\left[F_n\right]$

- so futures price process is a $\mathbb{Q}$-martingale!

Since $F_n = S_n$ also have

$$F_0 \; = \; \mathsf{E}_0^{\mathbb{Q}}\left[S_n\right]. \tag{8}$$

Calculating Futures Prices

Question 1: What property of the cash account did we use in deriving $F_0 = \mathsf{E}_0^Q[S_n]$?

As a result (8) only holds when $\mathbb{Q}$ is the EMM corresponding to taking the cash account as numeraire.

Question 2: Does the equation $F_0 = \mathsf{E}_0^{\mathbb{Q}}[S_n]$ change if the underlying security pays dividends?

(In that case can take S_i to be the ex-dividend price of the security at time i.)

Example: Futures and American Options

Example: Commodity price follows binomial model with $u = 1.03$, $d = .98$, $R = 1.01$ per period and no storage costs.

Check that risk-neutral probabilities of up- and down-moves are $q = .6$ and $1 - q = .4$, respectively.

Have the following price lattice:

Commodity Price

						119.41
					115.93	113.61
				112.55	110.30	108.09
			109.27	107.09	104.95	102.85
		106.09	103.97	101.89	99.85	97.85
	103.00	100.94	98.92	96.94	95.00	93.10
100.00	98.00	96.04	94.12	92.24	90.39	88.58
t=0	t=1	t=2	t=3	t=4	t=5	t=6

Example: Futures and American Options

Let S_k denote the commodity price at time k.

Futures contract on commodity exists and expires after six periods.

Futures price therefore satisfies $F_k \;=\; R^{6-k} S_k$.

Obtain following futures price lattice:

Futures Price

						119.41
					117.09	113.61
				114.81	111.40	108.09
			112.58	109.24	105.99	102.85
		110.40	107.12	103.94	100.85	97.85
	108.25	105.04	101.92	98.89	95.95	93.10
106.15	103.00	99.94	96.97	94.09	91.30	88.58
t=0	t=1	t=2	t=3	t=4	t=5	t=6

Example: Futures and American Options

Suppose we wish to price American put option on the futures contract.

- Option expires at same time as futures contract, i.e. after 6 periods.
- Strike $= \$105$.

Then obtain the price lattice below for the option.

Price of American Put Option on Futures Price

						0.00
					0.00	0.00
				0.00	0.00	0.00
			0.13	0.34	0.85	2.15
		0.50	1.05	2.15	4.15	7.15
	1.12	2.09	3.70	6.11	9.05	11.90
2.00	3.37	5.38	8.03	10.91	13.70	16.42
t=0	t=1	t=2	t=3	t=4	t=5	t=6

Foundations of Financial Engineering

Martingale Pricing: Forward Contracts

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Calculating Forwards Prices

Let G_0 = date $t = 0$ price of a forward contract for delivery of security at $t = n$.

Recall G_0 chosen so that contract initially worth zero. Therefore obtain

$$0 = \mathsf{E}_0^{\mathbb{Q}}\left[\frac{S_n - G_0}{B_n}\right]$$

so that

$$G_0 = \frac{\mathsf{E}_0^{\mathbb{Q}}\left[S_n/B_n\right]}{\mathsf{E}_0^{\mathbb{Q}}\left[1/B_n\right]}. \tag{9}$$

Note that (9) holds regardless of whether or not underlying pays dividends

- or coupons or storage costs (which may be viewed as negative dividends).

Dividends (or other intermediate cash-flows) influence G_0 through evaluation of $\mathsf{E}_0^{\mathbb{Q}}\left[S_n/B_n\right]$

- remember when there are dividends, it is the *deflated cumulative gains process* that is a Q-martingale.

When Do Forwards and Futures Prices Coincide?

Can now identify when forwards and futures price coincide.

Theorem: If B_n and S_n are $\mathbb{Q}$-independent, then $G_0 = F_0$. In particular, if interest rates are deterministic, then $G_0 = F_0$.

Corollary: In the binomial model with a constant (or deterministic) gross interest rate, R, we must have $G_0 = F_0$.

Exercise: Suppose interest rates are stochastic and positively "correlated" with movements in the underlying market. Would you expect F_0 to be greater than the forward price, G_0, or less than it? Justify your answer.