Foundations of Financial Engineering

Introduction to Forwards

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Introduction to Forwards

Definition: A forward contract on a security is a contract agreed upon at $t = 0$ to purchase or sell the security at date T for a price, F, that is specified at $t = 0$.

When forward contract established at $t = 0$, the forward price, F, is set so that initial value of contract is $f_0 = 0$.

At maturity value of contract is

$$f_T := \pm(S_T - F)$$

where S_T = time T value of underlying security.

Very important to realize there are two "prices" / "values" associated with a forward contract at time t:

- The forward price, F
- The value of the forward contract, f_t.

Note f_t is generally not equal to zero for $t > 0$.

Examples of Forward Contracts

Examples of forward contracts include:

- A forward contract for delivery, i.e. purchase, of a stock with maturity 6 months.
- A forward contract for delivery of a 9-month T-Bill with maturity 3 months. (Upon delivery, the T-Bill will have 9 months to maturity.)
- A forward contract for the sale of gold with maturity 1 year.
- A forward contract for delivery of 10m Euro (in exchange for dollars) with maturity 6 months.

Computing Forward Prices: Zero "Storage Costs"

First consider forward contracts on securities that can be "stored" at zero cost

- and we assume short selling is allowed.

Origin of term "stored" is that of forward contracts on commodities such as gold or oil that are costly to store.

However, can also use the term when referring to financial securities:

e.g. dividend paying stocks and coupon bonds are stored at negative cost.

Proposition: The arbitrage-free forward price, F, at $t = 0$ for delivery of that security at date T is given by

$$F \;=\; S/d(0,T) \tag{1}$$

where S is the current spot price of the security and $d(0,T)$ is the discount factor applying to the interval $[0,T]$.

Computing Forward Prices: Zero "Storage Costs"

Proof: We will construct an "arbitrage portfolio" if $F \neq S/d(0,T)$.

Case (i): $F < S/d(0,T)$**:**

Consider the portfolio that at date $t = 0$ is short one unit of the security, lends S until date T, and is long one forward contract.

Initial cost of this portfolio is 0 and it has a positive payoff, $S/d(0,T) - F$, at date T.

Case (ii): $F > S/d(0,T)$**:**

Construct the reverse portfolio and again obtain an arbitrage opportunity. □

A Forward on a Non-Dividend Paying Stock

Example: Consider a forward contract on a non-dividend paying stock that matures in 6 months.

Current stock price is $\$50$ and the 6-month interest rate is 4% per annum.

Assuming semi-annual compounding, discount factor is given by $d(0, .5) = 1/1.02 = 0.9804$.

Arbitrage-free forward price then satisfies

$$\begin{aligned} F &= S/d(0,T) \\ &= 50/0.9804 \; = = 51.0. \end{aligned}$$

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Forward Prices When Storage Costs Are Non-Zero

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Forward Prices When Storage Costs Are Non-Zero

Suppose now underlying security has non-zero storage costs.

Will assume a multi-period setting and that the security has a deterministic holding cost of $c(j)$ in period j

- payable at beginning of the period.

For a commodity, $c(j)$ will generally represent a true (and positive) holding cost.

For a stock or bond, $c(j)$ will be a negative cost and represent a dividend or coupon payment.

Proposition: The arbitrage-free forward price, F, for delivery of the security in M periods time satisfies

$$F = \frac{S}{d(0, M)} + \sum_{j=0}^{M-1} \frac{c(j)}{d(j, M)} \qquad (2)$$

where S is the current spot price of the security and $d(j, M)$ is the discount factor for borrowing / lending between dates j and M. □

Forward Prices When Storage Costs Are Non-Zero

Proof: Consider strategy of buying one unit of the security on the spot market at $t = 0$, and simultaneously entering a forward contract to deliver it at time T.

Cash-flow associated with this strategy is

$$(-S - c(0), -c(1), \; \ldots \;, -c(j), \; \ldots \;, -c(M-1), \; F)$$

and its present value must (why?) be equal to zero.

Since the cash-flow is deterministic we know how to compute its present value and we easily obtain (2).

E.G. Pricing a Forward Contract on a Bond

Consider a forward contract on a 4-year bond with maturity 1 year. Current value of bond is $\$1018.86$, face value is $\$1000$ and coupon rate is 10% per annum.

A coupon has just been paid and further coupons will be paid after 6 months and after 1 year, just prior to delivery.

Interest rates for 1 year out are flat at 8%.

Note that here the storage costs, i.e. the coupon payments, are paid at the end of the period, which has length 6 months.

Therefore need to adjust (2) slightly to obtain

$$F \;=\; \frac{S}{d(0,M)} \;+\; \sum_{j=0}^{M-1} \frac{c(j)}{d(j+1,M)}.$$

So forward price given by

$$F \;=\; \frac{1018.86}{d(0,2)} \;-\; \frac{50}{d(1,2)} \;-\; 50$$

where $d(0,2) = 1.04^{-2}$ and $d(1,2) = d(0,2)/d(0,1) = 1.04^{-1}$.

Value of a Forward Contract When $t > 0$

Recall that by construction the value of a forward contract satisfies $f_0 = 0$.

Proposition: Let F_t be the current forward price at date t for delivery of the same security at the same maturity date, T. We then have

$$f_t \; = \; (F_t - F_0)\; d(t, T). \qquad (3)$$

Proof: Consider a portfolio that at date t goes long one unit of a forward contract with price F_t and maturity T, and short one unit of a forward contract with price F_0 and maturity T.

This portfolio has a deterministic cash-flow of $F_0 - F_t$ at date T and a deterministic cash-flow of f_t at date t.

Present value at date t of this cash-flow stream, $(f_t,\; F_0 - F_t)$ must be zero (why?) and hence we obtain (3).

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Introduction to Swaps

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Plain Vanilla Interest Rate Swaps

- Have a maturity T, a notional principal P, and a fixed number of periods M.
- Two counter-parties: A and B.
- In each period party A makes a payment to party B corresponding to a fixed rate of interest on P.
- Similarly, in every period party B makes a payment to party A that corresponds to a floating rate of interest on the same notional principal, P.
- Note the principal itself is never exchanged.
- Must also specify whether payments occur at end or beginning of each period.

Plain Vanilla Interest Rate Swaps

e.g. Assume cash payments are made at the end of each period, i.e. in arrears.

Then total aggregate cash flow from party A's perspective is

$$C = P \times (0, \underbrace{r_0 - r_f}_{\text{At end of } 1^{st} \text{ period}}, \ldots, \underbrace{r_{M-1} - r_f}_{\text{At end of } M^{th} \text{ period}})$$

- $r_f =$ (constant) fixed rate
- $r_i =$ floating rate, i.e. the rate that prevailed at beginning of period i.

In general, r_i stochastic and so swap's cash-flow, C, also stochastic.

Fixed rate r_f usually chosen so that initial value of swap is zero.

Even though initial value is zero, say party A is "long" and party B is "short".

Pricing the Vanilla Interest-Rate Swap

The cash flow C, can be decomposed into:

1. A series of fixed payments, $P \times (0,\ r_f,\ r_f,\ \ldots\ ,\ r_f)$
    - easily priced.
2. A stochastic stream, $P \times (0,\ r_0,\ r_1,\ \ldots\ ,\ r_{M-1})$.

Can value the stochastic stream by noting / recalling that price of a floating rate bond is always par at any reset point:

Pricing the Vanilla Interest-Rate Swap

Note that stochastic stream is exactly the stream of coupon payments corresponding to a floating rate bond with face value P.

Hence value of stochastic stream must be (why?) $P(1 - d(0,M))$ and so value of swap is given by

$$V = P\left[1 - d(0,M) - r_f \sum_{i=1}^{M} d(0,i)\right]. \tag{4}$$

Pricing A Commodity Swap

Let $S_i =$ spot price of a commodity at beginning of period i.

Party A receives the spot price for N units of the commodity and pays a fixed amount, X, per period.

Will assume that payments take place at beginning of each period and there will be a total of M payments, beginning one period from now.

Cash-flow as seen by party that is long the swap is

$$C \;=\; N \times (0,\; S_1 - X,\; S_2 - X,\; \ldots\;,\; S_M - X)\,.$$

But C is stochastic and so cannot compute its present value directly by discounting.

Pricing A Commodity Swap

But can decompose C into:

1. A stream of fixed payments $-N \times (0,\ X,\ X,\ \ldots\ , X)$
   - easily priced.
2. A stochastic stream, $N \times (0,\ S_1,\ S_2,\ \ldots\ ,\ S_M)$.

But receiving $N \times S_i$ at period i has same value as receiving $N \times F_i$ at period i where $F_i =$ date 0 forward price for delivery of commodity at date i.

So stochastic stream easily seen to be equivalent to a stream of forward contacts on N units of the commodity.

Pricing A Commodity Swap

Since the F_i's are deterministic and known at date 0, value of commodity swap is given by

$$V \;=\; N\sum_{i=1}^{M} d(0,i)(F_i - X).$$

X is usually chosen so that the initial value of V is zero.

Foundations of Financial Engineering

Hedging with Futures

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A Perfect Hedge

Example: A wheat producer knows he will have $100,000$ bushels of wheat available to sell in three months time.

He is concerned the spot price of wheat will move against him, i.e. fall, in the meantime.

So he decides to lock in the sale price now by hedging in the futures markets.

Each wheat futures contract is for $5,000$ bushels, so he decides to sell 20 three-month futures contracts.

Note that as a result, the wheat producer has a perfectly hedged position.

Perfect Hedges

In general, perfect hedges are not available for a number of reasons:

1. None of the expiration dates of available futures contracts may exactly match the expiration date of the payoff, P_T, that we want to hedge.
2. P_T may not correspond exactly to an integer number of futures contracts.
3. The security underlying the futures contract may be different to the security underlying P_T.
4. P_T may be a non-linear function of the security price underlying the futures contract.
5. Combinations of all the above are also possible.

When perfect hedges are not available can use the minimum-variance hedge.

Constructing Minimum-Variance Hedges

Let Z_T = date T cash flow that we wish to hedge and let F_t = time t price of futures contract.

At date $t = 0$ we adopt a position of h in the futures contract and hold this position until time T.

Since initial cost of a futures position is zero, can write terminal cash-flow as

$$Y_T = Z_T + h(F_T - F_0).$$

Our objective then is to minimize

$$\text{Var}(Y_T) = \text{Var}(Z_T) + h^2\text{Var}(F_T) + 2h\text{Cov}(Z_T, F_T)$$

Constructing Minimum-Variance Hedges

Find that minimizing h and minimum variance are given by

$$\begin{aligned} h^* &= -\frac{\mathrm{Cov}(Z_T, F_T)}{\mathrm{Var}(F_T)} \\ \mathrm{Var}(Y_T^*) &= \mathrm{Var}(Z_T) - \frac{\mathrm{Cov}(Z_T,\ F_T)^2}{\mathrm{Var}(F_T)}. \end{aligned}$$

Such static hedging strategies are often used in practice

- but dynamic hedging strategies are capable of achieving a smaller variance.

Hedging Operating Profits

Example: A firm manufactures a particular type of widget and has orders to supply D_1 and D_2 of these widgets at dates t_1 and t_2, respectively.

The revenue, R, of the firm may then be written as

$$R = D_1 P_1 + D_2 P_2$$

where P_i represents the price per widget at time t_i.

P_i is stochastic and will depend in part on general state of economy at date t_i.

We assume

$$P_i = a S_i \, e^{\epsilon_i} + c$$

where a and c are constants, $S_i =$ time t_i value of market index, and ϵ_1 and ϵ_2 are independent random variables that are also independent of S_i.

Also assume that $\mathrm{E}[e^{\epsilon_i}] = 1$ for each i.

Hedging Operating Profits

The firm wishes to hedge the revenue, R, by taking a position h at $t = 0$ in a futures contract that expires at date t_2 and where the market index is the underlying security.

Ignoring the time value of money, the date t_2 payoff, Y, is then given by

$$Y \;=\; D_1\,(aS_1\;e^{\epsilon_1}\;+\;c)\;+\;D_2\,(aS_2\;e^{\epsilon_2}\;+\;c)\;+\;h(S_2 - F_0).$$

If S_t is a geometric Brownian motion so that

$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma B_t\big)$$

where B_t is a standard Brownian motion, then can easily find the minimum variance hedge, $h^* = -\mathsf{Cov}(R, S_2)/\mathsf{Var}(S_2)$.

Question: Can you construct more sophisticated hedging strategies?

Foundations of Financial Engineering

Introduction to Options

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Introduction to Options

Definition: A European call option gives the right but not the obligation to purchase 1 unit of the underlying security at a pre-specified price, K, at a pre-specified time T.

Definition: An American call option gives the right but not the obligation to purchase 1 unit of the underlying security at a pre-specified price K at any time up to and including a pre-specified time T.

Definition: A European put option gives the right but not the obligation to sell 1 unit of the underlying security at a pre-specified price K at a specified time T.

Definition: An American put option gives the right but not the obligation to sell 1 unit of the underlying security at a pre-specified price K at any time up to and including a pre-specified time T.

K and T are called the strike and maturity / expiration, respectively.

Payoff and Intrinsic Value of European Call and Puts

Payoff of a European call option at expiration $= \max\{S_T - K, 0\}$

Intrinsic value of a call option at time $t \leq T$ is $\max\{S_t - K, 0\}$

- In the money: $S_t > K$
- At the money: $S_t = K$
- Out of the money: $S_t < K$

Payoff of a European put option at expiration $= \max\{K - S_T, 0\}$

Intrinsic value of a put option at time $t \leq T$ is $\max\{K - S_t, 0\}$

- In the money: $S_t < K$
- At the money: $S_t = K$
- Out of the money: $S_t > K$

Options have nonlinear payoff so cannot price them without a model for the underlying security – but important results still available!

Put-Call Parity

Let $p_E(t; K, T)$ and $c_E(t; K, T)$ denote prices of European put and call.

Let $p_A(t; K, T)$ and $c_A(t; K, T)$ denote prices of American put and call.

Theorem: European put-call parity at time t for non-dividend paying stock:

$$p_E(t; K, T) + S_t = c_E(t; K, T) + Kd(t, T)$$

Proof: Consider following trading strategy:

- At time t buy European call with strike K and expiration T
- At time t sell European put with strike K and expiration T
- At time t (short) sell 1 unit of underlying and buy at time T
- At time t lend $d(t, T)K$ dollars until time T

Bounds on European Option Prices

Suppose underlying security does not pay dividends.

Suppose also the events $\{S_T > K\}$ and $\{S_T < K\}$ have strictly positive probability – a very reasonable assumption.

Can then use put-call parity to obtain

$$\begin{aligned} c_E(t; K, T) &= S_t + p_E(t; K, T) - Kd(t, T) \\ &> S_t - Kd(t, T). \end{aligned} \tag{5}$$

Consider now corresponding American call option:

Bounds on European Option Prices

So price of American call on a non-dividend paying stock always strictly greater than intrinsic value when events $\{S_T > K\}$ and $\{S_T < K\}$ have strictly positive probability

- have shown it's never optimal to early-exercise an American call on a non-dividend paying stock so $c_A(t; K, T) = c_E(t, K, T)$

No such result holds for American put options.

Foundations of Financial Engineering

The 1-Period Binomial Model

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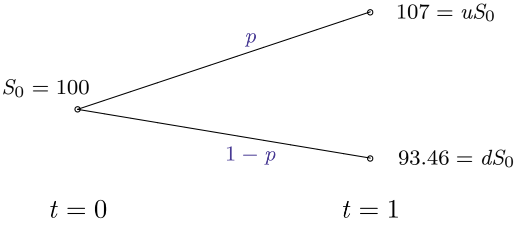
The 1-Period Binomial Model



- Can borrow or lend at gross risk-free rate, R
  - so \$1 in cash account at $t = 0$ is worth \$$R$ at $t = 1$
- Also assume that short-sales are allowed.

Type A and Type B Arbitrage

Need more general definitions of arbitrage when we introduce randomness.

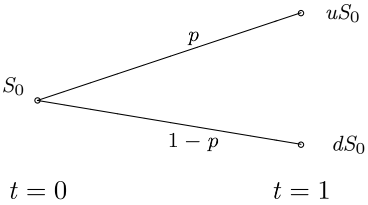
Definition: A type A arbitrage is a security or portfolio that produces immediate positive reward at $t = 0$ and has non-negative value at $t = 1$.

i.e. a security with initial cost, $V_0 < 0$, and time $t = 1$ value $V_1 \geq 0$.

Definition: A type B arbitrage is a security or portfolio that has a non-positive initial cost, has positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then.

i.e. a security with initial cost $V_0 \leq 0$, and $V_1 \geq 0$ but $V_1 \neq 0$.

Arbitrage in the 1-Period Binomial Model



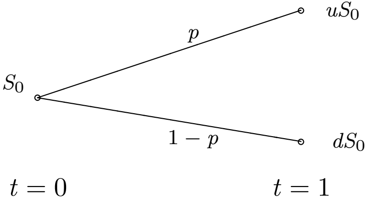
- Recall we can borrow or lend at gross risk-free rate, R, per period.
- And short-sales are allowed.

Theorem: There is no arbitrage if and only if $d < R < u$.

Proof: Suppose not the case that $d < R < u$. Then have 2 possibilities:

(i) $R \leq d < u$: Then at $t = 0$ borrow S_0 and buy the stock.

Arbitrage in the 1-Period Binomial Model



(ii) $d < u \leq R$: Then short-sell one unit of stock at $t = 0$ and invest proceeds in cash-account.

Will soon see other direction, i.e. if $d < R < u$, then there can be no-arbitrage.

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Option Pricing in the 1-Period Binomial Model

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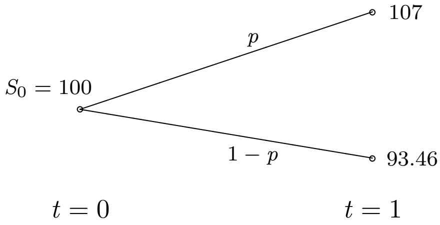
Option Pricing in the 1-Period Binomial Model



Assume now that $R = 1.01$.

1. How much is a call option that pays $\max(S_1 - 102,\ 0)$ at $t = 1$ worth?
2. How will the price vary as p varies?

To answer these questions, we will construct a replicating portfolio.

The Replicating Portfolio

- Consider buying x shares and investing $\$y$ in cash at $t = 0$
- At $t = 1$ this portfolio is worth:

$$\begin{array}{rl} 107x + 1.01y & \text{when } S = 107 \\ 93.46x + 1.01y & \text{when } S = 93.46 \end{array}$$

- Can we choose x and y so that portfolio equals option payoff at $t = 1$?
- If so, then we must solve

$$\begin{aligned} 107x + 1.01y &= 5 \\ 93.46x + 1.01y &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} x &= 0.3693 \\ y &= -34.1708 \end{aligned}$$

So yes, we can construct a replicating portfolio!

The Replicating Portfolio

Question: What does a negative value of y mean?

Question: What would a negative value of x mean?

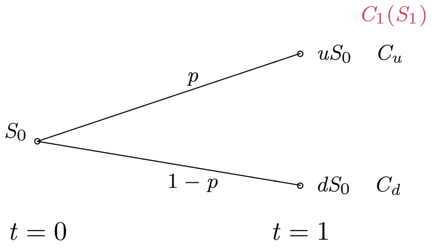
- The cost of this portfolio at $t = 0$ is

$$0.3693 \times 100 - 34.1708 \times 1 \approx 2.76$$

- So the fair value of the option is 2.76
  - indeed 2.76 is the arbitrage-free value of the option.

- Therefore option price does not directly depend on buyer's (or seller's) utility function or (apparently) the true probabilities, p and $1 - p$, of up- and down-moves, respectively.

Derivative Security Pricing



- Can use same replicating portfolio argument to find price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$.
- Set up replicating portfolio as before:

$$uS_0x + Ry = C_u \quad (6)$$
$$dS_0x + Ry = C_d \quad (7)$$

- Solve for x and y as before and then must have $C_0 = xS_0 + y$.

Derivative Security Pricing

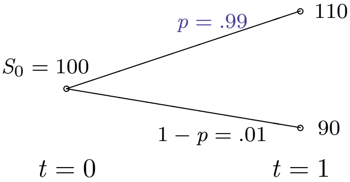
- After solving (6) and (7) can easily check(!)

$$\begin{aligned} C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\ &= \frac{1}{R}\left[q\,C_u + (1-q)\,C_d\right] \\ &= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1]. \end{aligned} \tag{8}$$

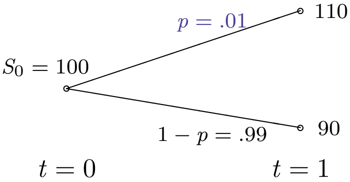
- Note that if $d < R < u$ then $q > 0$ and $1 - q > 0$ and (8) implies (why?) there can be no-arbitrage
  - we refer to (8) risk-neutral pricing
  - and $(q, 1 - q)$ are the risk-neutral probabilities.
- So we now know how to price any derivative security in this 1-period model.
- Can also answer earlier question: "How does the option price depend on p?"
  - but is the answer crazy?!

What's Going On?

- Stock ABC



- Stock XYZ



Question: How does the price of a call option on ABC with strike $K = \$100$ compare to the price of a call option on XYZ with strike $K = \$100$?

Foundations of Financial Engineering

The Multi-Period Binomial Model

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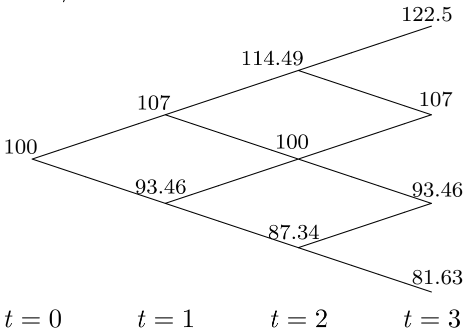
A 3-period Binomial Model

Let $R = 1.01$ and $u = 1/d = 1.07$.

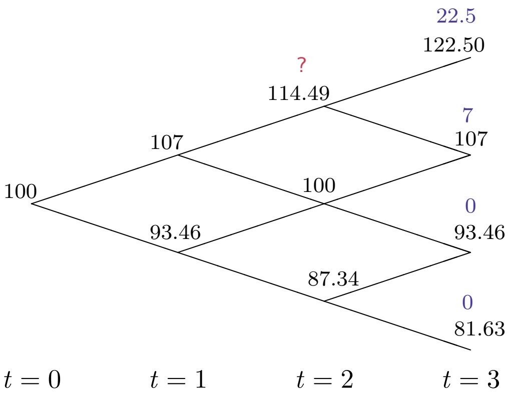


Just a series of 1-period models spliced together!

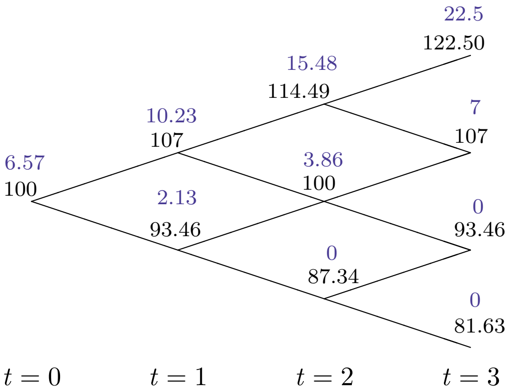
- all the results from the 1-period model apply
- just need to multiply 1-period probabilities along branches to get probabilities in multi-period model.

Pricing a European Call Option

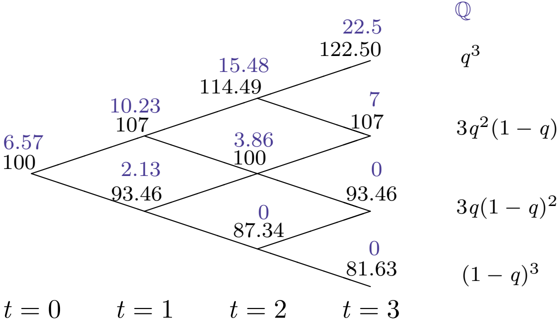
Assumptions: expiration at $t = 3$, strike $= \$100$ and $R = 1.01$.



Pricing a European Call Option



Pricing a European Call Option



But can also calculate the price directly:

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Pricing American Options

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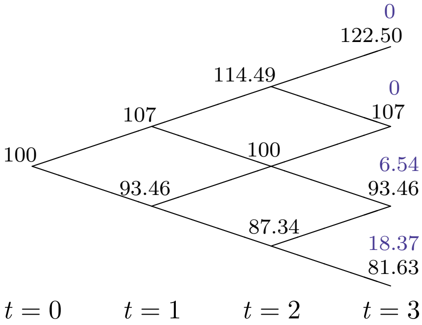
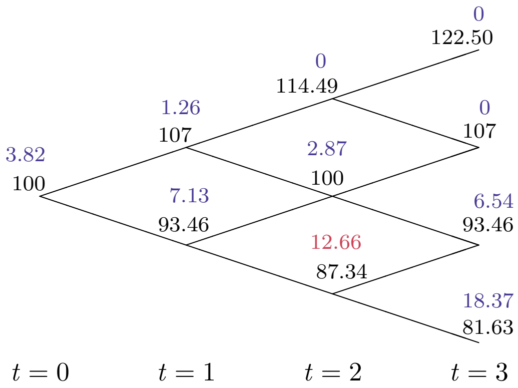
Pricing American Options

- Can also price American options in same way as European options
  - but must also check if it's optimal to early exercise at each node.
- But recall never optimal to early exercise an American call option on non-dividend paying stock.

Example: Price American put: expiration at $t = 3$, $K = \$100$ and $R = 1.01$.



Pricing American Options



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Including Dividends

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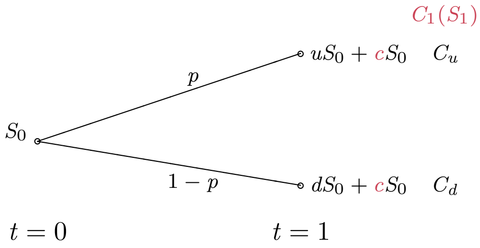
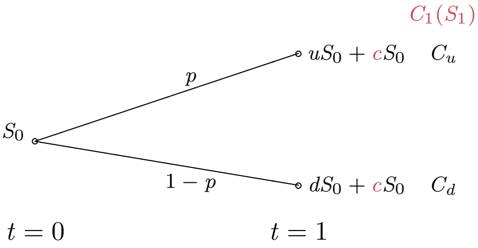
Including Dividends



- Consider again 1-period model and assume stock pays a proportional dividend of cS_0 at $t = 1$.
- No-arbitrage conditions are now $d + c < R < u + c$.

Including Dividends



- Can use same replicating portfolio argument to find price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$.
- Set up replicating portfolio as before:

$$uS_0x + cS_0x + Ry = C_u \quad (9)$$
$$dS_0x + cS_0x + Ry = C_d \quad (10)$$

- Solve for x and y as before and then must have $C_0 = xS_0 + y$.

Derivative Security Pricing with Dividends

- Solving (9) and (10) we obtain:

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d-c}{u-d}C_u + \frac{u+c-R}{u-d}C_d\right] \qquad (11)\\
&= \frac{1}{R}\left[q\,C_u + (1-q)\,C_d\right]\\
&= \frac{1}{R}\mathsf{E}_0^{\mathbb{Q}}[C_1].
\end{aligned}
$$

So once again, we can price any derivative security in this 1-period model with dividends!

Multi-Period Binomial Model with Dividends

Multi-period binomial model assumes a proportional dividend in each period

- so dividend of cS_i is paid at $t = i + 1$ for each i.

Then each embedded 1-period model has identical risk-neutral probabilities

- and derivative securities priced as before.

In practice dividends are not paid in every period

- and are therefore just a little more awkward to handle.