Foundations of Financial Engineering

The Black-Scholes Model

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The Black-Scholes Model

Will derive the Black-Scholes PDE for a call-option on a non-dividend paying stock with strike K and maturity T.

Assume stock price follows a GBM:

$$dS_t = \mu S_t \, dt \; + \; \sigma S_t \, dW_t \tag{1}$$

where W_t is a standard Brownian motion.

Also assume that continuously compounded interest rate is a constant, r

- so 1 unit invested in cash account at $t = 0$ worth $B_t := e^{rt}$ at time t.

By Itô's lemma know that

$$dC(S,t) = \left(\mu S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) \, dt \; + \; \sigma S_t \frac{\partial C}{\partial S} \, dW_t \tag{2}$$

- where we use $C(S,t)$ to denote time t call option price.

The Black-Scholes Model

Consider now a self-financing (s.f.) trading strategy where at each time t we hold x_t units of the cash account and y_t units of the stock.

Then time t value of this strategy is

$$P_t = x_t B_t + y_t S_t. \tag{3}$$

Will choose x_t and y_t so that the strategy replicates the value of the option.

The s.f. assumption implies

$$\begin{aligned} dP_t &= x_t \, dB_t \; + \; y_t \, dS_t & (4) \\ &= rx_t B_t \, dt + y_t \left(\mu S_t \, dt \; + \; \sigma S_t \, dW_t\right) \\ &= \left(rx_t B_t + y_t \mu S_t\right) \, dt \; + \; y_t \sigma S_t \, dW_t. & (5) \end{aligned}$$

Note that (4) is consistent with definition of s.f. in discrete-time models.

The Black-Scholes Model

Let's equate terms in (2) with corresponding terms in (5) to obtain

$$y_t = \frac{\partial C}{\partial S} \tag{6}$$

$$r x_t B_t = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}. \tag{7}$$

If we set $C_0 = P_0$ then must be the case that $C_t = P_t$ for all t since C and P now have identical dynamics.

Substituting (6) and (7) into (3) we obtain the Black-Scholes PDE:

$$rS_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{8}$$

In order to solve (8) boundary conditions must also be provided.

In the case of our call option those conditions are:

$C(S,T) = \max(S-K,0)$, $C(0,t) = 0$ for all t and $C(S,t) \to S$ as $S \to \infty$.

The Black-Scholes Solution

Solution to (8) (in the call option case) is

$$C(S,t) \;=\; S_t\Phi(d_1) \;-\; e^{-r(T-t)}K\Phi(d_2) \tag{9}$$

$$\text{where} \quad d_1 \;=\; \frac{\log\left(\frac{S_t}{K}\right) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\text{and} \quad d_2 \;=\; d_1 - \sigma\sqrt{T-t}$$

and $\Phi(\cdot)$ is the standard normal CDF.

One way to confirm (9) is to compute the various partial derivatives using (9), substitute them into (8) and check that (8) holds.

The price of a European put-option can also now be easily computed from put-call parity and (9).

The Black-Scholes Solution

The most interesting feature of the Black-Scholes PDE (8) is that μ does not appear anywhere!

- hence the name risk-neutral pricing.

Foundations of Financial Engineering

The Volatility Surface

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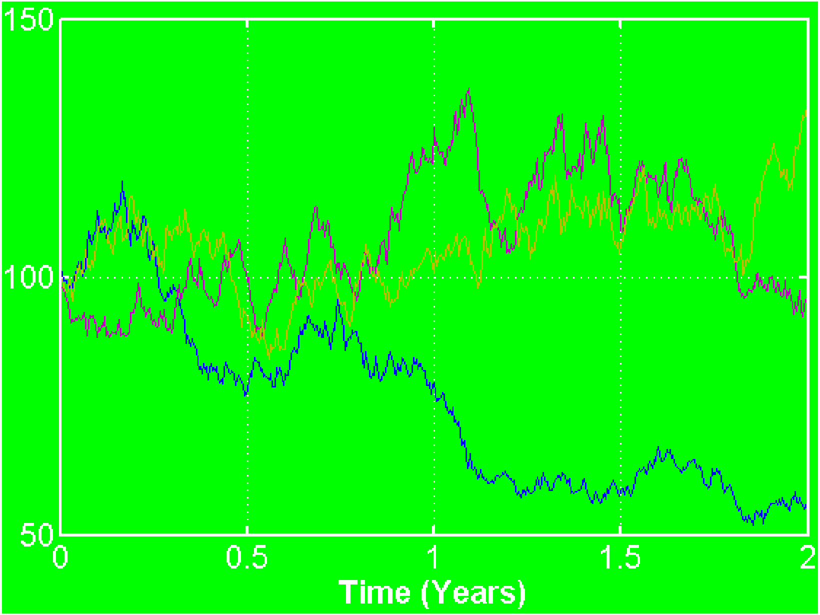
The Volatility Surface

The Black-Scholes model is elegant but it does not perform well in practice:

- Well known that stock prices can jump and do not always move in continuous manner predicted by GBM
- Stock prices also tend to have fatter tails than predicted by GBM.


150
100
50
0
0.5
1
1.5
2
Time (Years)

The Volatility Surface

If B-S model correct then should have a flat implied volatility surface.

The volatility surface defined implicitly by

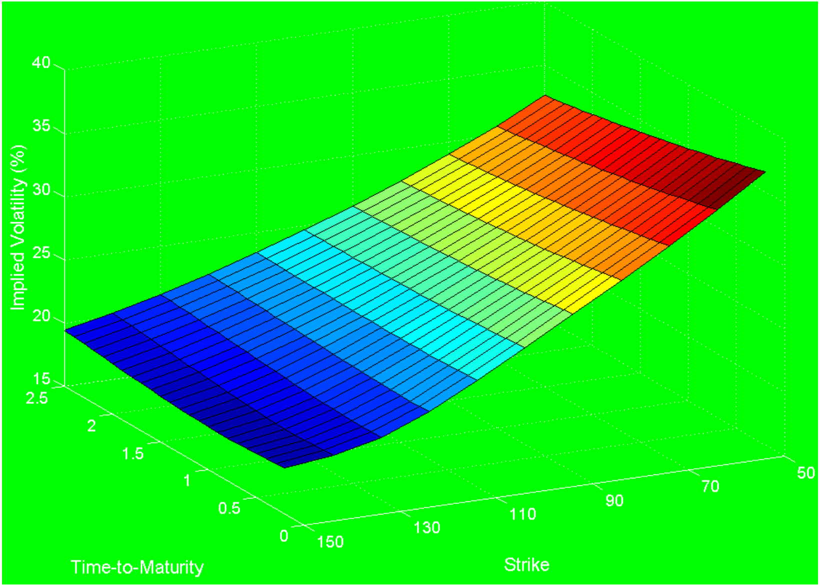
$$C(S, K, T) := \mathsf{BS}\,(S, T, r, q, K, \sigma(K, T)) \tag{10}$$

where $C(S, K, T) =$ current market price of call option and $\mathsf{BS}(\cdot) =$ B-S price.

There will always (why?) be a unique solution, $\sigma(K, T)$, to (10).

If B-S model correct then volatility surface would be flat with $\sigma(K, T) = \sigma$.

In practice, however, volatility surface is not flat and it actually moves randomly in time.



Implied Volatility (%)
40
35
30
25
20
15
2.5
2
1.5
1
0.5
0
Time-to-Maturity
150
130
110
90
70
50
Strike

Arbitrage Constraints on the Volatility Surface

Shape of volatility surface is constrained by absence of arbitrage requirement:

1. Must have $\sigma(K, T) \geq 0$ for all strikes K and expirations T.

2. At any given maturity, T, the skew cannot be too steep
   - otherwise put spread arbitrage will exist.

   To see this, fix a maturity T and consider put options with strikes $K_1 < K_2$.

   If no arbitrage then must be the case (why?) that $P(K_1) < P(K_2)$.

   However, if skew is too steep then would obtain (why?) $P(K_1) > P(K_2)$.

Calendar Spread Arbitrage

3. Likewise the term structure of implied volatility cannot be too inverted
   - otherwise calendar spread arbitrages will exist.

Most easily seen in the case where $r = q = 0$.

So fix a strike K and let $C_t(T)$ denote time t price of a call option with strike K and maturity T.

Martingale pricing then implies S_t is a $\mathbb{Q}$-martingale and $(S_t - K)^+$ is a $\mathbb{Q}$-"submartingale".

"Standard" martingale results imply $C_t(T) = \mathsf{E}_t^{\mathbb{Q}}[(S_T - K)^+]$ must be non-decreasing in T

- would be violated (why?) if term structure of implied volatility too inverted.

Foundations of Financial Engineering

Why is there a Skew?

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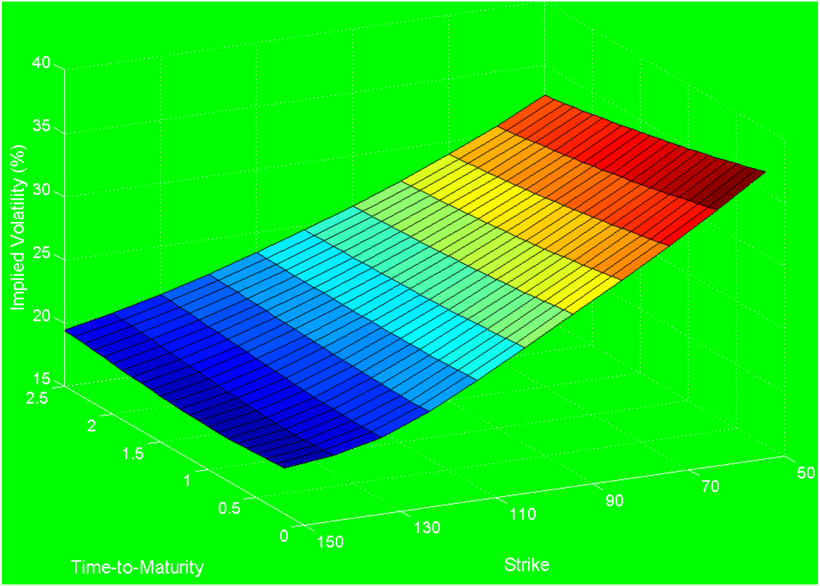
Implied Volatility (%)
40
35
30
25
20
15
2.5
2
1.5
1
0.5
0
Time-to-Maturity
150
130
110
90
70
50
Strike

Why is there a Skew?

For stocks and stock indices there is generally a skew so that for any fixed maturity, T, the implied volatility decreases with the strike, K.

Most pronounced at shorter expirations for two reasons:

1. Risk aversion – can appear in many guises:
   (i) Stocks do not follow GBM but instead often jump. Jumps to downside tend to be larger and more frequent than jumps to upside.
   (ii) As markets go down, fear sets in and volatility goes up.
   (iii) Supply and demand: investors like to protect their portfolio by purchasing OTM puts so there is more demand for options with lower strikes.

2. The leverage effect. Based on fact that total value of company assets is a more natural candidate to follow GBM.

   In this case equity volatility should increase as equity value decreases.

The Leverage Effect

Let V, E and D denote value of firm, firm's equity and firm's debt.

Then fundamental accounting equation states $V = D + E$.

Let ΔV, ΔE and ΔD be change in values of V, E and D.

Then $V + \Delta V = (E + \Delta E) + (D + \Delta D)$ so that

$$\begin{aligned} \frac{V + \Delta V}{V} &= \frac{E + \Delta E}{V} + \frac{D + \Delta D}{V} \\ &= \frac{E}{V}\left(\frac{E + \Delta E}{E}\right) + \frac{D}{V}\left(\frac{D + \Delta D}{D}\right). \end{aligned} \tag{11}$$

The Leverage Effect

If equity component is substantial so that debt is not too risky, then (11) implies

$$\sigma_V \approx \frac{E}{V}\,\sigma_E$$

where σ_V and σ_E are the firm value and equity volatilities.

Therefore have

$$\sigma_E \approx \frac{V}{E}\,\sigma_V. \qquad (12)$$

The Leverage Effect

Example: Suppose $V = 1$, $E = .5$ and $\sigma_V = 20\%$

- then (12) implies $\sigma_E \approx 40\%$.

Suppose σ_V remains unchanged but that firm loses 20% of its value over time.

Almost all of this loss is borne by equity so (12) now implies $\sigma_E \approx 53\%$.

σ_E has therefore increased despite the fact that σ_V has remained constant!

Remark: There was little or no skew in market before Wall Street crash of 1987

- but then the market woke up to the flaws of GBM!

Foundations of Financial Engineering

What the Volatility Surface Tells Us

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What the Volatility Surface Tells Us

Recall the volatility surface is constructed from European option prices.

Consider a butterfly strategy centered at K where you are:

1. long a call option with strike $K - \Delta K$
2. long a call with strike $K + \Delta K$
3. short 2 call options with strike K

Value of butterfly at time $t = 0$ is

$$
\begin{aligned}
B_0 &= C(K - \Delta K, T) - 2C(K, T) + C(K + \Delta K, T) \\
&\approx e^{-rT} \,\text{Prob}(K - \Delta K \leq S_T \leq K + \Delta K) \times \Delta K/2 \\
&\approx e^{-rT} \, f(K, T) \times 2\Delta K \times \Delta K/2 \\
&= e^{-rT} \, f(K, T) \times (\Delta K)^2
\end{aligned}
$$

where $f(K, T)$ is the (risk-neutral) PDF of S_T evaluated at K.

What the Volatility Surface Tells Us

Therefore have

$$f(K,T) \quad \approx \quad e^{rT} \; \frac{C(K-\Delta K, T) - 2C(K,T) + C(K+\Delta K, T)}{(\Delta K)^2}. \qquad (13)$$

Letting $\Delta K \to 0$ in (13), we obtain

$$f(K,T) = e^{rT} \; \frac{\partial^2 C}{\partial K^2}.$$

Volatility surface therefore gives the marginal risk-neutral distribution of the stock price, S_T, for any time, T.

It tells us nothing about the joint distributions of the stock price at multiple times $(S_{T_1}, \ldots, S_{T_n})$

- not surprising as volatility surface is constructed from European option prices and they only depend on marginal distributions of S_T.

Same Marginals But Different Joint Distributions

There are two times, T_1 and T_2, of interest and a non-dividend paying security A has risk-neutral dynamics that satisfy

$$S_{T_1}^A = e^{(r-\sigma^2/2)\,T_1+\sigma\sqrt{T_1}\,Z_1^A} \quad (14)$$

$$S_{T_2}^A = e^{(r-\sigma^2/2)\,T_2+\sigma\sqrt{T_2}\,\left(\rho_A Z_1^A+\sqrt{1-\rho_A^2}Z_2^A\right)} \quad (15)$$

where Z_1^A and Z_2^A are independent $\mathsf{N}(0,1)$ random variables.

A value of $\rho_A > 0$ can capture a momentum effect and a value of $\rho_A < 0$ captures a mean-reversion effect.

Suppose now there is another non-dividend paying security B with risk-neutral distributions given by

$$S_{T_1}^B = e^{(r-\sigma^2/2)\,T_1+\sigma\sqrt{T_1}\,Z_1^B} \quad (16)$$

$$S_{T_2}^B = e^{(r-\sigma^2/2)\,T_2+\sigma\sqrt{T_2}\,\left(\rho_B Z_1^B+\sqrt{1-\rho_B^2}Z_2^B\right)} \quad (17)$$

where Z_1^B and Z_2^B are again independent $\mathsf{N}(0,1)$ random variables.

Same Marginals But Different Joint Distributions

Observation: If Z_1 and Z_2 are independent N(0,1) random variables then for any $\rho \in [-1, 1]$

$$\rho Z_1 + \sqrt{1-\rho^2} Z_2 \sim \mathsf{N}(0, 1)$$

Therefore see that:

$S^A_{T_1}$ and $S^B_{T_1}$ have the same marginal risk-neutral distributions.

$S^A_{T_2}$ and $S^B_{T_2}$ have the same marginal risk-neutral distributions.

Therefore follows that options on A and B with same strike and maturity must have same price

- so A and B have identical "volatility surfaces".

Same Marginals But Different Joint Distributions

Example: Now consider a knock-in put option with strike 1 and expiration T_2.

In order to "knock-in", stock price at time T_1 must exceed barrier price of 1.2.

Payoff function therefore given by

$$\text{Payoff} \;=\; \max\left(1 - S_{T_2},\; 0\right)\; 1_{\{S_{T_1} \geq 1.2\}}.$$

Question: Would the knock-in put option on A have the same price as the knock-in put option on B?

Same Marginals But Different Joint Distributions

Question: How does your answer depend on ρ_A and ρ_B?

Question: What does this say about the ability of the volatility surface to price barrier options?

Foundations of Financial Engineering

The Greeks: Delta

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Put-Call Parity

The "Greeks" measure the sensitivity of the option price to changes in various parameters.

The Greeks are usually computed using the B-S formula
- despite fact that B-S model known to be a poor approximation to reality.

But first recall put-call parity:

$$e^{-rT} K + \text{ Call Price } = e^{-qT} S + \text{ Put Price.} \tag{18}$$

where call and put have same strike K, and maturity T, and $q =$ dividend yield.

Put-call parity very useful for:

1. Calculating Greeks. e.g. it implies that Vega(Call) $=$ Vega(Put)
2. For calibrating dividends or borrow rate
3. Constructing the volatility surface.

The Greeks: Delta

Definition: The delta of an option is the sensitivity of the option price to a change in the price of the underlying security.

Delta of a European call option (in B-S model) is

$$\text{delta} \;=\; \frac{\partial C}{\partial S} \;=\; e^{-qT}\,\Phi(d_1).$$

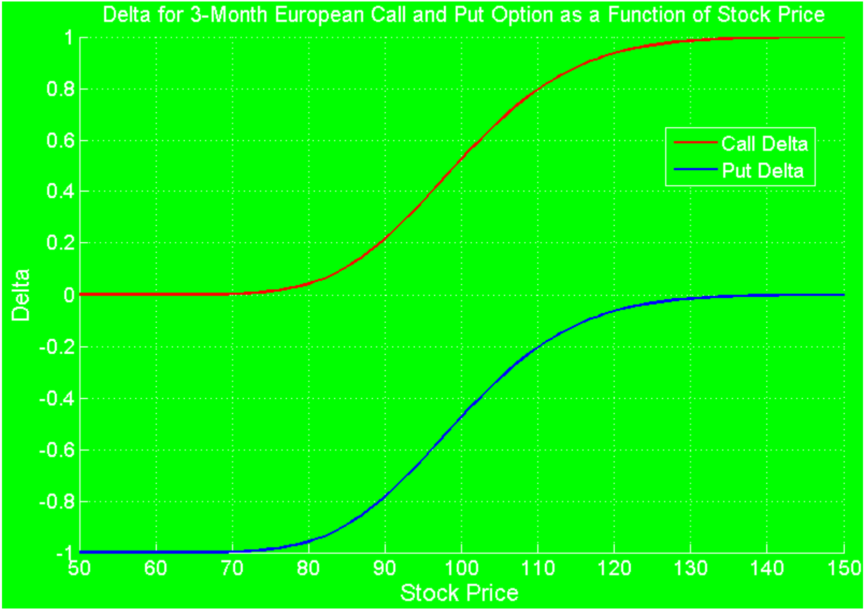
– the "usual" delta corresponding to a volatility surface that is sticky-by-strike

- assumes volatility of option does not move when underlying price moves.

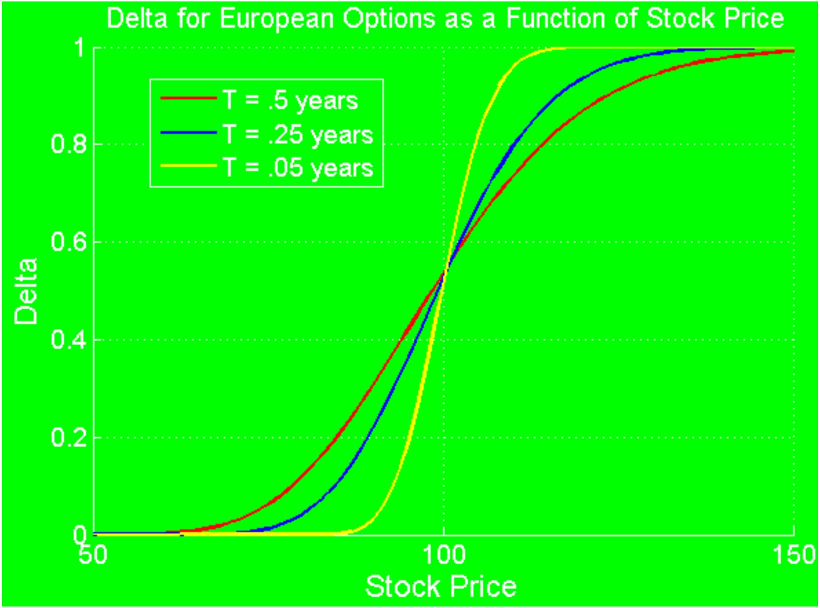
If volatility of option did move then delta would have an additional term of the form vega $\times\, \partial\sigma(K,T)/\partial S$

- in this case would say that the volatility surface was sticky-by-delta.

In following figures we assumed $r = q = 0$ and $K = 100$.



Delta for 3-Month European Call and Put Option as a Function of Stock Price
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Delta
50
60
70
80
90
100
110
120
130
140
150
Stock Price
Call Delta
Put Delta



Delta for European Options as a Function of Stock Price
T = .5 years
T = .25 years
T = .05 years
Delta
0
0.2
0.4
0.6
0.8
1
50
100
150
Stock Price

The Greeks: Delta

By put-call parity, have

$$\text{delta}_{\text{put}} = \text{delta}_{\text{call}} - e^{-qT}.$$

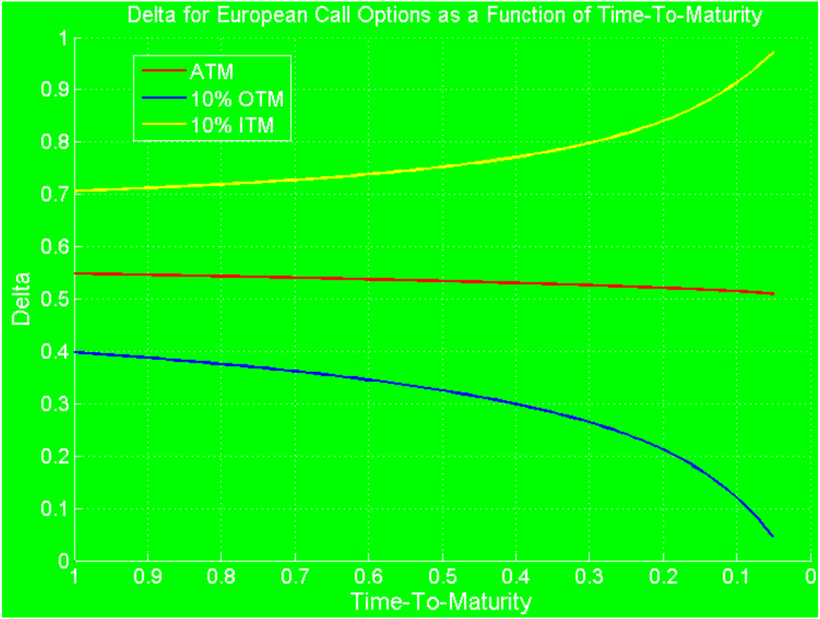
Note that delta becomes steeper around K when time-to-maturity decreases.

Note also that

$$\text{delta}_{\text{call}} = \Phi(d_1)$$

is often interpreted as (risk-neutral) probability of option expiring in the money

- this probability is in fact equal to $\Phi(d_2)$.


Delta for European Call Options as a Function of Time-To-Maturity
ATM
10% OTM
10% ITM
Delta
Time-To-Maturity
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Foundations of Financial Engineering

The Greeks: Gamma

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The Greeks: Gamma

Definition: The gamma of an option is the sensitivity of the option's delta to a change in the price of the underlying security.

The gamma of a call option satisfies

$$\text{gamma} = \frac{\partial^2 C}{\partial S^2} = e^{-qT} \frac{\phi(d_1)}{\sigma S \sqrt{T}}$$

where $\phi(\cdot)$ is the standard normal PDF.

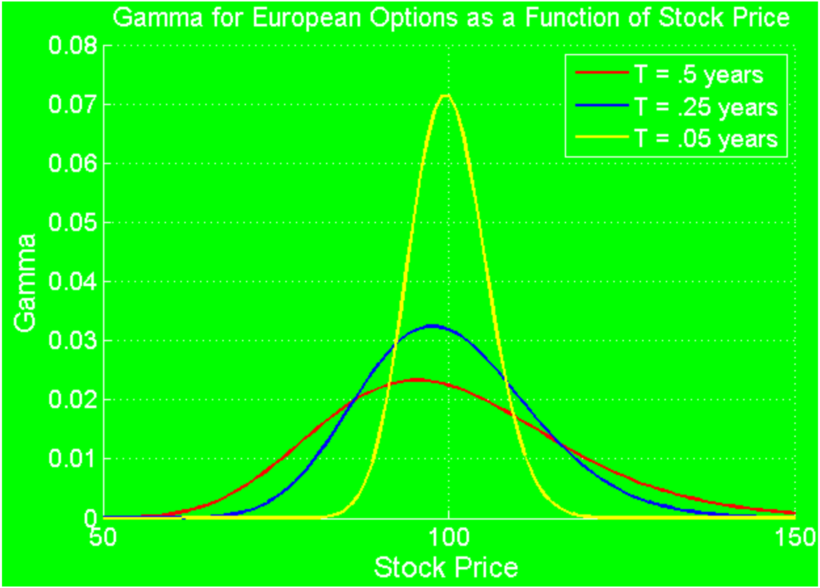
By put-call parity, gamma of European call = gamma of European put with same strike and maturity.

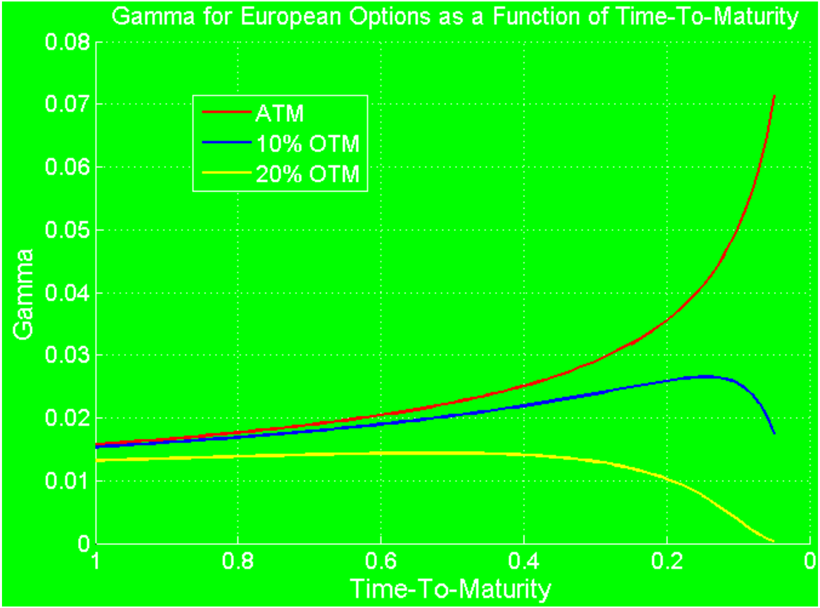
Gamma always positive due to option convexity and traders who are long gamma can make money by gamma scalping

- process of regularly re-balancing option portfolio to be delta-neutral.

However, must pay for this long gamma position with the option premium!

Gamma Scalping


Gamma for European Options as a Function of Stock Price
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Gamma
50
100
150
Stock Price
T = .5 years
T = .25 years
T = .05 years


Gamma for European Options as a Function of Time-To-Maturity
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Gamma
ATM
10% OTM
20% OTM
1
0.8
0.6
0.4
0.2
0
Time-To-Maturity

Foundations of Financial Engineering

The Greeks: Vega

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The Greeks: Vega

Definition: The vega of an option is the sensitivity of the option price to a change in volatility.

The vega of a call option satisfies

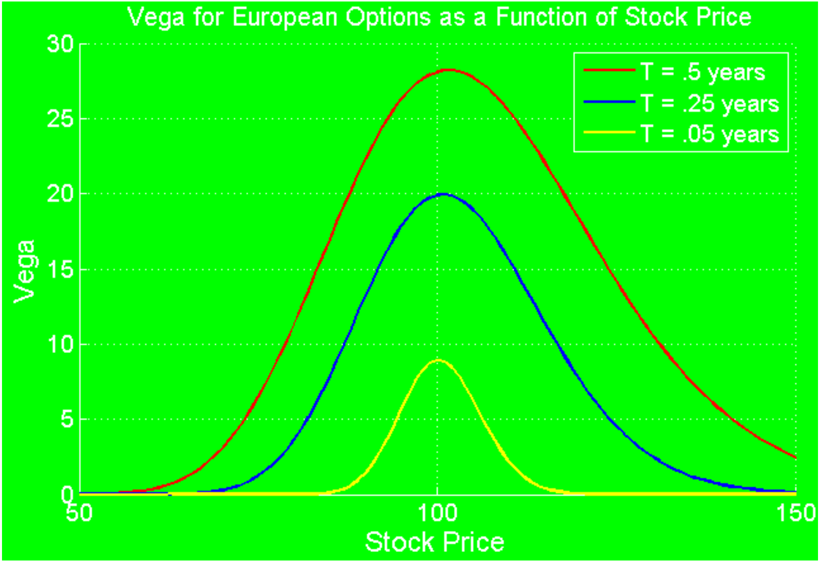
$$\text{vega} = \frac{\partial C}{\partial \sigma} = e^{-qT} S \sqrt{T}\, \phi(d_1).$$

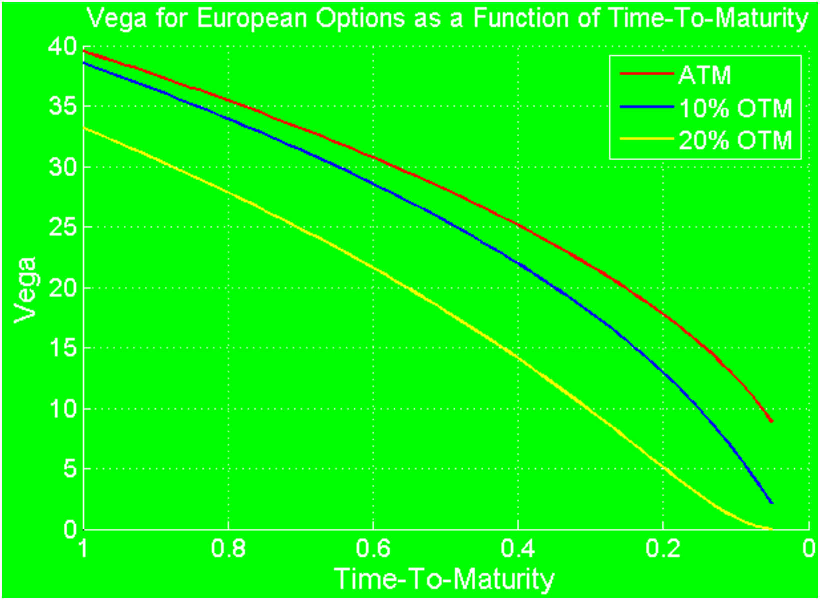
Put-call parity implies vega of European call = vega of European put with same strike and maturity.

In following figures we assumed $K = 100$ and that $r = q = 0$.

Question: Why does vega increase with time-to-maturity?

Question: For a given time-to-maturity, why is vega peaked near the strike?


Vega for European Options as a Function of Stock Price
30
25
20
15
10
5
0
Vega
50
100
150
Stock Price
T = .5 years
T = .25 years
T = .05 years


Vega for European Options as a Function of Time-To-Maturity
ATM
10% OTM
20% OTM
Vega
Time-To-Maturity
40
35
30
25
20
15
10
5
0
1
0.8
0.6
0.4
0.2
0

Foundations of Financial Engineering

The Greeks: Theta

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The Greeks: Theta

Definition: The theta of an option is the sensitivity of the option price to a negative change in time-to-maturity.

The theta of a call option satisfies

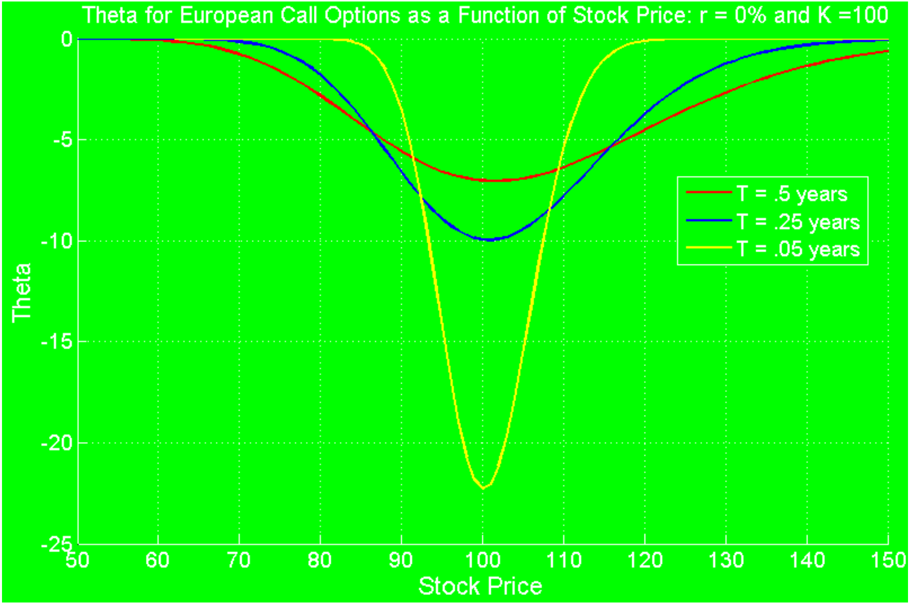
$$\text{theta} = -\frac{\partial C}{\partial T} = -e^{-qT} S \phi(d_1) \frac{\sigma}{2\sqrt{T}} + q e^{-qT} S \Phi(d_1) - rKe^{-rT} \Phi(d_2).$$

In following figures have assumed $r = q = 0\%$ and $K = 100$.

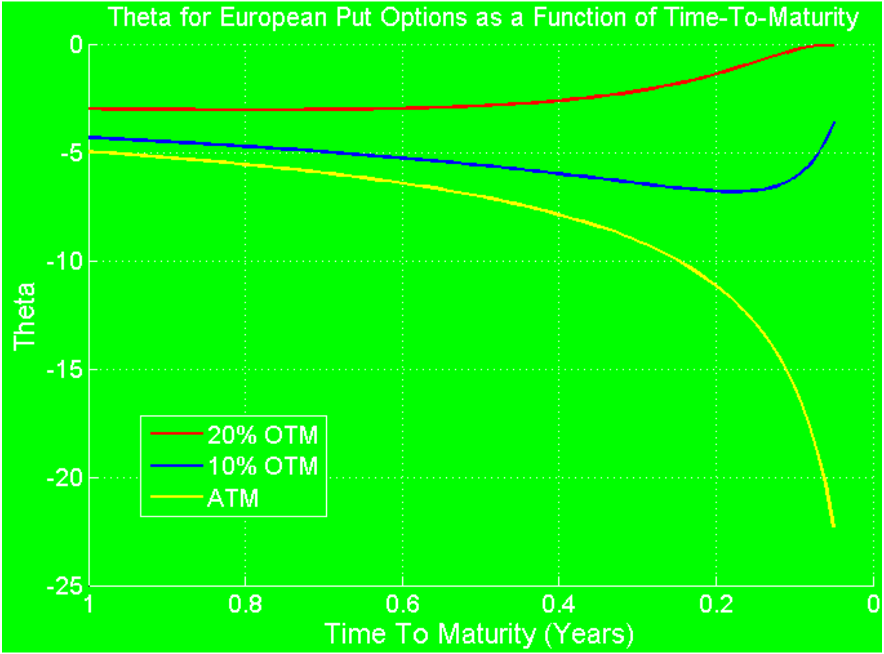
Note that call option's theta is always negative (in these figures).

Can you explain why this is the case?

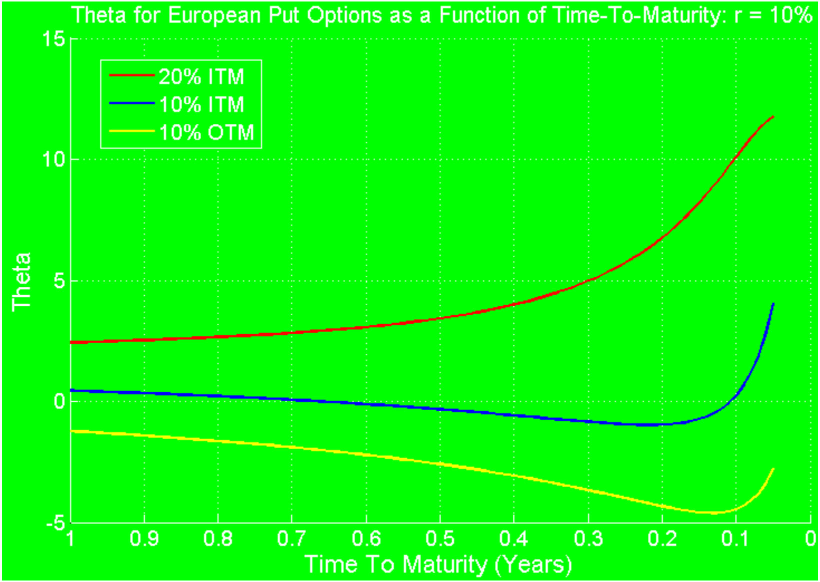
Question: Why does theta become more negatively peaked as time-to-maturity decreases?



Theta for European Call Options as a Function of Stock Price: r = 0% and K =100
0
-5
-10
-15
-20
-25
Theta
50
60
70
80
90
100
110
120
130
140
150
Stock Price
T = .5 years
T = .25 years
T = .05 years



Theta for European Put Options as a Function of Time-To-Maturity
0
-5
-10
-15
-20
-25
Theta
1
0.8
0.6
0.4
0.2
0
Time To Maturity (Years)
20% OTM
10% OTM
ATM



Still have $q = 0$ but now $r = 10\%$. Note theta positive for ITM put. Why?

Can also obtain positive theta for call options when q is large.

Foundations of Financial Engineering

Delta-Gamma-Vega Approximations to Option Prices

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Delta-Gamma-Vega Approximations to Option Prices

A simple application of Taylor's Theorem yields

$$\begin{aligned} C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\ &= C(S,\sigma) + \Delta S \times \delta + \frac{1}{2}(\Delta S)^2 \times \Gamma + \Delta\sigma \times \text{vega} \end{aligned}$$

where $C(S,\sigma) =$ price of a derivative security as a function of S and σ.

Therefore obtain

$$\begin{aligned} \text{P\&L} &= \delta \Delta S + \frac{\Gamma}{2}(\Delta S)^2 + \text{vega } \Delta\sigma \\ &= \text{delta P\&L} + \text{gamma P\&L} + \text{vega P\&L} \end{aligned}$$

Delta-Gamma-Vega Approximations to Option Prices

When $\Delta\sigma = 0$, obtain the well-known delta-gamma approximation

- often used in historical Value-at-Risk (VaR) calculations for portfolios that include options.

Can also write

$$\begin{aligned} \text{P\&L} &= \delta S \left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\ &= \text{ESP} \times \text{Return} + \$ \text{ Gamma} \times \text{Return}^2 + \text{vega } \Delta\sigma \end{aligned}$$

where ESP denotes the equivalent stock position or “dollar” delta.

Foundations of Financial Engineering

Delta-Hedging

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Delta-Hedging

Delta-hedging is act of re-balancing a portfolio continuously so that always have a total "delta" of zero

- in fact we obtained B-S PDE via a delta-hedging / replication argument.

Not practical of course to hedge continuously

- so instead we hedge periodically – results in some replication error.

Let P_t = time t value of discrete-time s.f. strategy that attempts to replicate the option payoff.

Let C_0 = initial value of the option.

Delta-Hedging

Replicating strategy then given by

$$P_0 := C_0 \tag{19}$$

$$P_{t_{i+1}} = P_{t_i} + \left(P_{t_i} - \delta_{t_i} S_{t_i}\right) r\Delta t + \delta_{t_i}\left(S_{t_{i+1}} + qS_{t_i}\Delta t - S_{t_i}\right) \tag{20}$$

- $\Delta t := t_{i+1} - t_i$ is the length of time between re-balancing
- $r =$ annual risk-free interest rate (assuming per-period compounding)
- δ_{t_i} is the B-S delta at time t_i
- q is the dividend yield.

Note that (19) and (20) respect the s.f. condition.

Delta-Hedging

Recall δ_{t_i} satisfies

$$\delta_{t_i} = \frac{\partial C}{\partial S} \;=\; e^{-q(T-t)}\;\Phi(d_1)$$

where

$$d_1 := \frac{\log\left(\frac{S_{t_i}}{K}\right) + (r + \sigma_{imp}^2/2)(T - t_i)}{\sigma_{imp}\sqrt{T - t_i}}$$

Stock prices are simulated assuming $S_t \sim \mathsf{GBM}(\mu, \sigma)$ so that

$$S_{t+\Delta t} \;=\; S_t\, e^{(\mu - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\, Z}$$

where $Z \sim \mathsf{N}(0, 1)$.

Note that σ_{imp} need not equal σ!

This has interesting implications for trading P&L which we define as

$$\text{P\&L} \;:=\; P_T - (S_T - K)^+$$

in the case of a short position in a call option with strike K and maturity T.

Delta-Hedging

Many interesting questions now arise:

Question: If you sell options, what typically happens the total P&L if $\sigma < \sigma_{imp}$?

Question: If you sell options, what typically happens the total P&L if $\sigma > \sigma_{imp}$?

Question: If $\sigma = \sigma_{imp}$ what typically happens the total P&L as the number of re-balances increases?

Recall that fair price of an option increases as the volatility increases.

Therefore if $\sigma > \sigma_{imp}$ we expect to lose money on average when we delta-hedge an option that we sold.

Similarly, we expect to make money when we delta-hedge if $\sigma < \sigma_{imp}$.

In general the payoff from delta-hedging an option is path-dependent.

Some Answers to Delta-Hedging Questions

Can be shown that payoff from continuously delta-hedging an option satisfies

$$\text{P\&L} = \int_0^T \frac{S_t^2}{2} \frac{\partial^2 C_t}{\partial S^2} \left(\sigma_{imp}^2 - \sigma_t^2\right) \, dt \tag{21}$$

where σ_t is the realized instantaneous volatility at time t.

Recall the dollar gamma term $\frac{S_t^2}{2} \frac{\partial^2 C_t}{\partial S^2}$:

- always positive for a call or put option
- but it goes to zero as the option moves significantly into or out of the money.

Returning to s.f. trading strategy of (19) and (20), note that we can choose any model we like for the security price dynamics

- interesting to see what happens when we depart from GBM dynamics!