IEOR E4706 Foundations of Financial Engineering
Martin Haugh — Due: 5pm Wednesday 23^{rd} November 2016

Assignment 8 (Mandatory)

1. **(Pricing a CDS in a Structural Model)**

   (a) Replicate the equity and debt lattices on page 3 of the *Credit Modeling and Credit Derivatives* lecture notes. These lattices give the price processes for the debt and equity components of the capital structure under the Black-Cox default framework. (You will of course need to first construct the firm value lattice as given on page 2 of the lecture notes.)

   (b) Consider a CDS on the debt component of part (a) where the premium leg makes a payment in every period and the default leg pays the difference between the face value of the bond, i.e. \$800, and the firm value upon default. Use separate lattices to compute the time $t = 0$ value of the premium and default legs. Now use trial and error to determine the fair dollar premium per period.

2. **(Estimating Ratings Transitions)**
   Consider the ratings models as described in Section 2 of the *Credit Modeling and Credit Derivatives* lecture notes. Given empirical transitions data, how should we estimate, for example, the 1-year transition matrix $\mathbf{P}$? There are a few approaches and we will consider two of them here. In order to simplify matters let us assume that there are just four possible ratings, A, B, C and D (default). In addition we have the following data from the past year:

   - At the beginning of the year, there were 10 firms in each of the four classes.
   - In the 1^{st} month, one B-rated firm migrated to a C-rating and one B-rated firm migrated to an A-rating.
   - In the 2^{nd} month, two A-rated firms migrated to a B-rating.
   - In the 6^{th} month, one C-rated firm migrated to a B-rating and one C-rated firm migrated to a D-rating.
   - In the 8^{th} month, one C-rated firm migrated to a D-rating.
   - There were no other transitions during the year.

   In the following questions you can assume that transitions took place at the *end* of the month in question.

   (a) What is $\widehat{\mathbf{P}}_{emp}$, the empirical estimator of $\mathbf{P}$ that we obtain by simply counting the empirical transitions between ratings in the given data-set?

(b) An alternative estimator is based on the so-called *generator* matrix, Λ, of the Markov chain. The generator matrix for a Markov chain with n states has the form

$$\Lambda = \begin{pmatrix} -\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\ \lambda_{2,1} & -\lambda_{2,2} & \cdots & \lambda_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n,1} & \lambda_{n,2} & \cdots & -\lambda_{n,n} \end{pmatrix}.$$

Note that each row in Λ sums to 0. Moreover, $\lambda_{i,j}$ for $i \neq j$, represents the transition rate from state i to state j. We can estimate it according to

$$\hat{\lambda}_{i,j} = \frac{M_{i,j}}{N_i}$$

where N_i is the total number (across all firms) of time units spent in state i and $M_{i,j}$ is the total number of transitions from i to j. Using the data-set given above, we would estimate

$$\hat{\lambda}_{A,B} = \frac{2}{8 \times \frac{12}{12} + 2 \times \frac{2}{12} + 1 \times \frac{11}{12}}. \tag{1}$$

Note that the time unit here is "years" and a term such as $\frac{2}{12}$ in the denominator of (1) corresponds to 2 months $= \frac{2}{12}$ of a year. The final piece we need is a well-known and very important property of Λ is that it can be used to determine the m-step transition matrix for any m. In particular, we have

$$\mathbf{P}^m = \exp\{\Lambda m\}$$

where exp denotes the *matrix exponential* function. This function is defined according to

$$\exp\{\mathbf{A}\} := \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!}$$

and it can be evaluated quickly using diagonalization methods. (You don't need to do that, however, as the `Matlab` function *expm* will calculate $\exp\{\mathbf{A}\}$ for you.)

Estimate $\mathbf{P}$ using the methodology / theory described here.

(c) Which of the methods in (a) and (b) yields a superior estimator in your opinion? Justify your answer.

3. **(Intensity Calculations)**
We can define the default intensity, λ_t to be

$$\lambda_t := \frac{f_\tau(t)}{\mathrm{P}(\tau > t)}$$

where τ is the time of default and $f_\tau(\cdot)$ is the PDF of τ. It therefore follows that $\lambda_t \, dt$ is equal to the probability of defaulting in the interval $(t, t + dt)$ given that default has not occurred in $[0, t]$. Show that

$$\mathrm{P}(\tau > s) = \exp\left\{ - \int_0^s \lambda_t \, dt \right\}.$$

4. **(Constructing Intensity-Based Lattice Models)**
   Reproduce the three lattices on page 10 of the *Credit Modeling and Credit Derivatives* lecture notes. (The details are described in Section 4.1 of the notes.)

5. **(Pricing a CDS in a Lattice Intensity Model)**
   Compute the fair spread of a CDS on the bond of Question 4 except now you should assume that the recovery upon default is 40% of face value. To simplify calculations you should assume that the premium leg makes a payment every period, i.e. year, and that the default leg pays the par amount, i.e. the face value of $100, minus the recovery upon default.

   Hint: See the final paragraph of Section 4.1 of the *Credit Modeling and Credit Derivatives* lecture notes to see how the fair spread can be calculated very quickly.