IEOR E4706 Foundations of Financial Engineering
Martin Haugh Due: 5pm Thursday $20^{th}$ October 2016

**Assignment 5**

1. **(A 2-Fund Theorem)**
Consider the mean-variance problem formulation when there is no risk-free asset available:

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

$$\text{subject to} \quad \mathbf{w}'\mu = p \tag{1}$$
$$\text{and} \quad \mathbf{w}'\mathbf{1} = 1. \tag{2}$$

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be (mean-variance) efficient portfolios corresponding to expected returns $r_1$ and $r_2$, respectively, with $r_1 \neq r_2$. Show that **all** efficient portfolios can be obtained as linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$.

2. **(Portfolio Benchmarks; Luenberger Q6.8)**
Suppose that it it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given efficient portfolio). One alternative is to find the portfolio, made up of a given set of $n$ stocks, that tracks the specified portfolio most closely – in the sense of minimizing the variance of the difference in returns.

Specifically, suppose that the target portfolio has (a random) rate of return $r_M$. Suppose that there are $n$ assets with (random) rates of return $r_1, \ldots, r_n$. We wish to find the portfolio rate of return

$$r = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n$$

(with $\sum_{i=1}^n \alpha_i = 1$) minimizing $\text{Var}(r - r_M)$.

(a) Find a set of equations for the optimal $\alpha_i$'s.

(b) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense – say, tracking efficient. Find the equation for the $\alpha_i$'s that are tracking efficient.

3. **(Project Evaluation; Luenberger Q7.8)**
Electron Wizards, Inc (EWI) has a new idea for producing TV sets, and it is planning to enter the development stage. Once the product is developed (which will be at the end of 1 year), the company expects to sell its new process for a price $p$, with expected value $\bar{p} = \$24\text{M}$. However, this sale price will depend on the market for TV sets at the time. By examining the stock histories of various TV companies, it is determined that the final sales price $p$ is correlated with the market rerum as $\text{E}[(p - \bar{p})(r_M - \bar{r}_M)] = \$20\text{M}\sigma_M^2$.

To develop the process, EWI must invest in a research and development project. The cost $c$ of this project will be known shortly after the project is begun (when a technical uncertainty will be resolved). The current estimate is that the cost will be either $c = \$20\text{M}$ or $c = \$16\text{M}$, and each of these is equally likely. (This uncertainty is uncorrelated with the final price and is also uncorrelated with the market.) Assume that the risk-free rate is $r_f = 9\%$ and the expected return on the market is $r_M = 33\%$.

(a) What is the expected rate of return of this project?

(b) What is the beta of this project? *Hint:* In this case, note that

$$\mathrm{E}\left[\left(\frac{p-\bar{p}}{c}\right)(r_M - \bar{r}_M)\right] = \mathrm{E}\left[\frac{1}{c}\right]\mathrm{E}\left[(p-\bar{p})(r_M - \bar{r}_M)\right].$$

(c) Is this an acceptable project based on a CAPM criterion? In particular, what is the excess rate of rerum (+ or -) above the return predicted by the CAPM?

4. **(A Leveraged Firm; Luenberger Q7.13)**
A company earns a rate of return of $r_A$ and has beta $\beta_A$. A fraction $w$ of the assets is owned by bondholders, and the remaining fraction $(1-w)$ is owned by equity holders. Every year the bondholders demand a riskless rate of return of $r_B$ on their fraction of the assets, regardless of the actual rate of return $r_A$ that was achieved that year. Beyond that, the equity holders take whatever is left after the bondholders have been paid.

(a) What is the rate of return of the equity holders in terms of $w$, $r_A$ and $r_B$?

(b) What is the beta of the rate of return of the equity holders in terms of $w$ and $\beta_A$?

(c) Suppose $\beta_A$ is positive and the expected rate of return on the market is greater than the risk-free rate. As $w$ increases (that is, as the firm becomes more leveraged), what should happen to the expected rate of return on the equity of the firm?

5. **(Linear Pricing with the CAPM)**
Show that the CAPM is consistent with linear pricing.

6. **(Estimation Errors)**
Download the `Matlab` program *MarkowitzAnalysis.m* from `Canvas`.

(a) Run it several times and make sure you understand what the program does. (You will need to have `Matlab`'s *Financial Toolbox* installed as some functions from this toolbox are used in the program. If you don't have access to the toolbox then you may either write your own code to implement these functions or you may skip this question!)

(b) Now edit the code so that it also:

i. Computes the *realized* frontier in addition to the true and estimated frontiers.

ii. Takes a parameter $n$ and on a single figure plots the true frontier, $n$ estimated frontiers and the $n$ corresponding realized frontiers. You should submit such a plot as apart of your assignment submission.

iii. What do you notice when you run your code for different values of the *NumSimSamples* variable?